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$\aleph_0$-Categoricity is not inherited by factor groups

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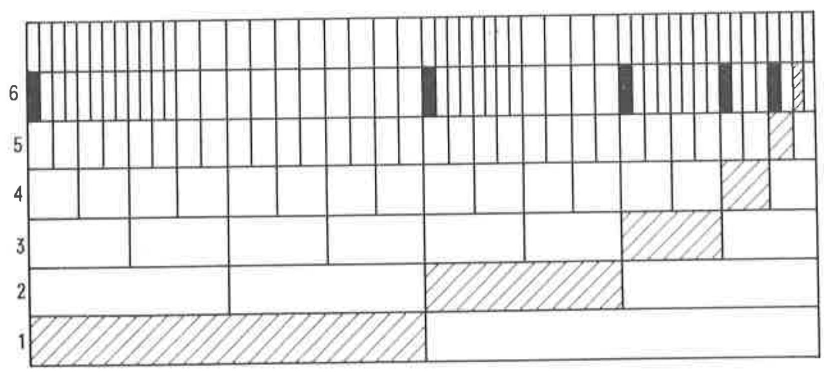
In the closing paragraph of [1], Olin cites a remark of R. Burns to the effect that if G is an $\aleph_0$-categorical group and $N \lhd G$ then G/N is also $\aleph_0$-categorical. This result was questioned by U. Felgner, who pointed out that it is true if H is a characteristic subgroup of G. Olin observed that it is also true (by the results of [2]) if G is Abelian.

In this note, we provide an example which shows that it is not true in general. The counterexample we present arises from the same groups considered by Olin in Example 3.3 of [1]; in that example, Olin showed that certain groups proved $\aleph_0$-categorical in [2] have subgroups which are not $\aleph_0$-categorical. We show here that those same groups also have factor groups which are not $\aleph_0$-categorical.

We start with a fixed sequence of partitions of the natural numbers ω. The m'th partition in the sequence has 2^m members, all of which are infinite, and each of its members is split into two members of the $(m+1)$'st partition. In the diagram below, the m'th level contains the 2^m members of the m'th partition, which, reading from left to right, we denote $S^m(1)$, $S^m(2), \ldots, S^m(2^m)$.



Now if $\mathscr{A}$ is a group, the D_0-limit direct power of $\mathscr{A}$, denoted $\mathscr{A}^\omega \mid D_0$, consists of those functions in $\mathscr{A}^\omega$ which for some m are constant on all the members of the m'th partition. Thus structurally we consider $\mathscr{A}^\omega \mid D_0$ as a subgroup of the full product $\mathscr{A}^\omega$. It follows from the work of Waskiewicz and Węgłorz (for references, see [1] or [2]) that if $\mathscr{A}$ is $\aleph_0$-categorical then so is $\mathscr{A}^\omega \mid D_0$.

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We define two subgroups H and K of $G=\mathscr{A}^{\omega} \mid D_0$. Let H be the collection of elements of G which are constant on each of the striped sections

$$S^1(1), \quad S^2(3), \quad S^3(7), \ldots, S^m(2^m-1), \ldots$$

of the diagram above. Let K be the collection of elements $g \in H$ which for some level $M(g)$ are equal to the identity on every $S^m(2^m-1)$ for $m \geqslant M(g)$.

Clearly H and K are subgroups of G and $K \lhd H$. The group K is isomorphic to $\oplus_{\omega}\mathscr{A}$ and is Olin's Example 3.3. (Define σ: $\oplus_{\omega}\mathscr{A} \to K$ by setting $\sigma(a)$ equal to a_m on $S^m(2^m-1)$ for $a=(a_m)$.) If $\mathscr{A}$ is finite but not Abelian, then, by [2], $\oplus_{\omega}\mathscr{A}$ is not $\aleph_0$-categorical, so that K is a non-$\aleph_0$-categorical subgroup of the $\aleph_0$-categorical group G.

We shall show that H is not $\aleph_0$-categorical if $\mathscr{A}$ is finite but not Abelian and that H is a homomorphic image of G. It follows that a factor group of an $\aleph_0$-categorical group need not be $\aleph_0$-categorical.

To see that H is not $\aleph_0$-categorical we observe that if $h \in H$ and $g \in K$ then there is an element $c \in K$ such that $h^{-1}gh=c^{-1}gc$ – namely, define c to equal h on $S^1(1)$, $S^2(3), \ldots, S^m(2^m-1)$ where $m=M(g)$ and equal to the identity on all $S^n(2^n-1)$ for $n>M(g)$. Thus each element of K has precisely the same number of conjugates in H as it has in K. Since the proof that K is not $\aleph_0$-categorical (see [2]) involves showing that there are elements of K which have arbitrary large, although finite, numbers of conjugates, the same is true of H and hence H is not $\aleph_0$-categorical.

To see that H is a homomorphic image of G we proceed as follows (referring to the solid sections on level 6 of the diagram.) Suppose $B \in G$ and suppose that B is constant on all the members of the m'th partition; define $\pi(B) \in H$ so that

(1) $\pi(B)$ on $S^1(1)$ is B on $S^m(1)$;
(2) $\pi(B)$ on $S^2(3)$ is B on $S^m(1+2^{m-1})$;
(3) $\pi(B)$ on $S^3(7)$ is B on $S^m(1+2^{m-1}+2^{m-2})$;
...
$(m-1)$ $\pi(B)$ on $S^{m-1}(2^{m-1}-1)$ is B on $S^m(1+2^{m-1}+\cdots+2^2)=S^m(2^m-3)$;
(m) $\pi(B)$ on $S^m(2^m-1)$ is B on $S^m(2^m-1)$

and

$$\pi(B) \text{ on } S^m(2^m) \text{ is } B \text{ on } S^m(2^m).$$

This definition of $\pi(B)$ is independent of which m is chosen, so long as B is constant on all members of the m'th partition, and defines a homomorphism of G to H. Since π is the identity on H, H is a homomorphic image of G.

Note. After submitting this counterexample we learned that G. Sabbagh has observed [4] that $(S_3)^{\omega} \mid D_0$ has non-$\aleph_0$-categorical factor groups. For by [3], $(S_3)^{\omega} \mid D_0$ is $GL_2(B)$ where B is a countable atomless Boolean ring and $GL_2(B')$ is a factor group of $GL_2(B)$ which is not $\aleph_0$-categorical if B' is atomic.

REFERENCES

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