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$\aleph_0$-categoricity of linear orderings

by

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Let M be an interpretation of a particular first-order language. The theory of M, $T(M)$, is the set of all statements in this language which are true in M. We say that M is *$\aleph_0$-categorical* if $T(M)$ is $\aleph_0$-categorical, i.e., every countable model of $T(M)$ is isomorphic to M.

Engeler [1], Ryll-Nardzewski [4], and Svenonius [5] gave a characterization of $\aleph_0$-categorical theories by taking a close look at certain Boolean algebras associated with a theory T. More specifically, if T is a theory we define $F_n(T)$ to be the set of well-formed formulas whose free variables are among $x_1, \ldots, x_n$. In $F_n(T)$ we introduce an equivalence relation by defining $\varphi \sim \psi$ if $\vdash_T (x_1)\ldots(x_n)(\varphi \equiv \psi)$. The equivalence classes then form a Boolean algebra with respect to the connectives $\wedge$, $\vee$, $\neg$; this Boolean algebra is denoted by $B_n(T)$. The theorem referred to above states that T is $\aleph_0$-categorical iff $B_n(T)$ is finite for each n.

In this note we shall improve this result in the case that T is an extension of the theory of linear orderings, and at the same time give a characterization of those countable linear orderings which are $\aleph_0$-categorical. More specifically, we define, similarly to Erdös and Hajnal [2] or Läuchli and Leonard [3], a set $\mathcal{M}$ of countable linear order types for which the following theorem holds:

THEOREM. *The following are equivalent:*

(i) $[M] \in \mathcal{M}$,

(ii) *M is $\aleph_0$-categorical,*

(iii) *$B_2(T(M))$ is finite.*

Let M be a linear ordering; we will also use M to mean the underlying set of M. The order relation on M will be denoted by $<$ (since there will be no danger of confusion.) A subset M_1 of M is called a *segment* if from $a \in M_1$, $b \in M_1$, and $a < c < b$ it follows that $c \in M_1$. An ordered set N is a splitting of M if N is a set of segments of M which partitions M and if $M_1 <_N M_2$ iff $a < b$ whenever $a \in M_1$ and $b \in M_2$. The elements of N are called the *parts* of M (relative to N.) If N and N^1 are splittings of M, then N is called a *refinement* of N^1 if every part of M relative to N^1 is contained in some part of M relative to N.

Let F be a finite non-empty set of order types. Suppose that there is a splitting of M of type η (the rationals) such that each part of the splitting has its order type in F and such that between any two parts there are parts having each of the order types in F. In this case we note that the order type of M is determined by the set F; it is denoted by σF (σ for "shuffle").

Let $\mathcal{M}$ be the smallest set of linear order types containing 1 and closed under $+$ and σ. The theorem stated above refers to this set.

Proof of the Theorem.

(i) $\Rightarrow$ (ii). We show by induction on the construction of $\mathcal{M}$ that if $[M] \in \mathcal{M}$ then M is $\aleph_0$-categorical.

I. $[M] = 1$. Since M is finite, it is $\aleph_0$-categorical.

II. $[M] = [M_1] + [M_2]$ where M_1 and M_2 are $\aleph_0$-categorical by induction hypothesis. Extend the language of linear orderings by adding two one-place relation symbols R_1 and R_2. Let T^* consist of the following statements of this language:

(1) $T(M)$,

(2) $(x)\big(R_1(x) \vee R_2(x)\big)$,

(3) $(x)\big\{\neg\big(R_1(x) \wedge R_2(x)\big)\big\}$,

(4) $(x)(y)\big(R_1(x) \wedge R_2(y) \Rightarrow x < y\big)$,

(5) $\{\varphi^{R_1} \mid \varphi \in T(M_1)\}$,

(6) $\{\varphi^{R_2} \mid \varphi \in T(M_2)\}$,

where, as usual, φ^R is φ with all quantifiers relativized to R. Then T^* is clearly consistent and $\aleph_0$-categorical. Hence $B_n(T^*)$ is finite for each n. We want to conclude that $B_n\big(T(M)\big)$ is finite for each n. But if φ, $\psi \in F_n\big(T(M)\big)$ and $\vdash_{T^*} (x_1)\ldots(x_n)(\varphi \equiv \psi)$ then, since $T(M)$ is complete, we must have $\vdash_{T(M)} (x_1)\ldots(x_n)(\varphi \equiv \psi)$. It follows that $B_n\big(T(M)\big)$ is finite for each n.

III. $[M] = \sigma F$ where $F = \{[M_1], [M_2], \ldots, [M_k]\}$ consists of order types of $\aleph_0$-categorical linear orderings $M_1, M_2, \ldots, M_k$. Extend the language of linear orderings by adding k one-place relation symbols $R_1, R_2, \ldots, R_k$. Let T^* consist of the following statements of this language:

(1) $T(M)$,

(2) $(x)\big(R_1(x) \vee R_2(x) \vee \ldots \vee R_k(x)\big)$,

(3) $(x)\big\{\neg\big(\bigvee_{1 \leqslant i < j \leqslant k} R_i(x) \wedge R_j(x)\big)\big\}$,

(4) $$(x)(y)\Big[x < y \wedge (\mathrm{E}z)\big(x < z < y \wedge \bigvee_{1 \leqslant i \leqslant k} \big\{R_i(z) \wedge \neg\big(R_i(x) \wedge R_i(y)\big)\big\}\big). \Rightarrow$$
$$\Rightarrow . \bigwedge_{1 \leqslant i \leqslant k} (\mathrm{E}z)\big(x < z < y \wedge R_i(z)\big)\Big],$$

$$(5) \quad \bigcup_{1\leqslant i\leqslant k} \{(x)(R_i(x) \Rightarrow \varphi^{c_x}) \mid \varphi \in T(M_i)\},$$

where it is understood that the variable x does not occur in φ and φ^{c_x} is the relativization of φ to $c_x(y)$:

$$\Big[x \leqslant y \wedge \bigwedge_{1\leqslant i\leqslant k} \big(R_i(x) \Rightarrow (z)(x \leqslant z \leqslant y \Rightarrow R_i(z))\big)\Big] \vee \\ \vee \Big[x \geqslant z \geqslant y \wedge \bigwedge_{1\leqslant i\leqslant k} \big(R_i(x) \Rightarrow (z)(x \geqslant z \geqslant y \Rightarrow R_i(z))\big)\Big].$$

Then T^* is clearly consistent and $\aleph_0$-categorical. Hence $B_n(T^*)$ is finite for all n. As we saw above this implies that $B_n(T(M))$ is finite for all n.

(ii) $\Rightarrow$ (iii). This follows from the theorem quoted earlier.

(iii) $\Rightarrow$ (i). Let M be a linear ordering for which $B_2(T(M))$, and hence $B_1(T(M))$, is finite. We shall, intuitively speaking, define a sequence of splittings of $\mathcal{M}$, each a refinement of the previous one, such that each part of each splitting has its order type in $\mathcal{M}$ and such that the final splitting will be of order type 1. From this we deduce that $[M] \in \mathcal{M}$.

More precisely, we define for each n a wff $C_n(x, y)$, which is satisfied by a pair $a \leqslant b$ of elements of M iff they are in the same part of the nth splitting, and a set X^n of wffs with one free variable (such that each element of M satisfies exactly one element of X^n) which encode the splitting history of elements of M.

Stage 0: $\varphi^0(x)$: $x = x$,

$$\Phi^0 = \{\varphi^0\}, \quad \Psi^0 = \emptyset, \quad \Theta^0 = \emptyset; \quad X^0 = \Phi^0 \cup \Psi^0 \cup \Theta^0,$$
$$C_0(x, y)\colon\ x = y.$$

Stage $m+1$: Let $X^m = \{X_1^m, X_2^m, \ldots, X_r^m\}$. For each finite sequence $t = \langle t_1, t_2, \ldots, t_s\rangle$, $s \geqslant 2$, of elements of $\{1, 2, \ldots, r\}$ define a wff $\varphi_t^{m+1}(x)$ by:

$$\varphi_t^{m+1}(x)\colon\ (\mathrm{E}x_1)(\mathrm{E}x_2)\ldots(\mathrm{E}x_s)\Big[\Big(\bigwedge_{1\leqslant i<s} x_i < x_{i+1}\Big) \wedge \Big(\bigvee_{1\leqslant i\leqslant s} x = x_i\Big) \wedge \Big(\bigwedge_{1\leqslant i\leqslant s} X_{t_i}^m(x_i)\Big) \wedge \\ \wedge (y)\Big(x_1 \leqslant y \leqslant x_s \Rightarrow \bigvee_{1\leqslant i\leqslant s} \big(C_m(x_i, y) \vee C_m(y, x_i)\big)\Big) \wedge \Big(\neg \bigvee_{1\leqslant i<s} C_m(x_i, x_{i+1})\Big) \wedge \\ \wedge (z)\big(z < x_1 \wedge \neg C_m(z, x_1). \Rightarrow .(\mathrm{E}w)(z < w < x_1 \wedge \neg C_m(z, w) \wedge \neg C_m(w, x_1))\big) \wedge \\ \wedge (z)\big(x_s < z \wedge \neg C_m(x_s, z). \Rightarrow .(\mathrm{E}w)(x_s < w < z \wedge \neg C_m(x_s, w) \wedge \neg C_m(w, z))\big)\Big].$$

For each subset $\{t_1, t_2, \ldots, t_s\}$ of $\{1, 2, \ldots, r\}$ define a wff $\psi_t^{m+1}(x)$ by:

$$\psi_t^{m+1}(x)\colon\ (\mathrm{E}y)(\mathrm{E}z)\Big[(y < x < z) \wedge (w)\Big(y < w < z \Rightarrow \bigvee_{1\leqslant i\leqslant s} X_{t_i}^m(w)\Big) \wedge \\ \wedge (c)(d)\Big(y \leqslant c < d \leqslant z \wedge \neg C_m(c, d). \Rightarrow . \\ \bigwedge_{1\leqslant i\leqslant s} (\mathrm{E}v)\big(c < v < d \wedge \neg C_m(c, v) \wedge \neg C_m(v, d) \wedge X_{t_i}^m(v)\big)\Big)\Big].$$

Let

$$\Phi^{m+1} = \{\varphi_i^{m+1}(x)|\ \text{for some}\ a \in M,\ M \models \varphi_i^{m+1}(a)\},$$

$$\Psi^{m+1} = \{\psi_i^{m+1}(x)|\ \text{for some}\ a \in M,\ M \models \psi_i^{m+1}(a)\}.$$

Note that these sets are finite.

For each j, $1 \leqslant j \leqslant r$, define a wff $\theta_j^{m+1}(x)$ by

$$\theta_j^{m+1}(x):\ X_j^n(x) \wedge \big(\neg \bigvee_{\varphi \in \Phi^{m+1}} \varphi(x)\big) \wedge \big(\neg \bigvee_{\psi \in \Psi^{m+1}} \psi(x)\big).$$

Let

$$\Theta^{m+1} = \{\theta_j^{m+1}(x)|\ \text{for some}\ a \in M,\ M \models \theta_j^{m+1}(a)\}$$

and let

$$X^{m+1} = \Phi^{m+1} \cup \Psi^{m+1} \cup \Theta^{m+1};$$

then X^{m+1} is a finite set of wffs.

Finally define $C_{m+1}(x, y)$ to be

$$x \geqslant y \wedge \bigvee_{\varphi \in X^{m+1}} (z)\big(x \leqslant z \leqslant y \Rightarrow \varphi(z)\big).$$

Each of the following is then easy to verify:

(i) *Every element of M satisfies exactly one wff of X^m.*

(ii) $S_a^m = \{b|\ C_m(a, b) \vee C_m(b, a)\}$ *is a segment of M for each a.*

(iii) $\mathbf{C}_m = \{S_a^m|\ a \in M\}$ *is a splitting of M which refines* $\mathbf{C}_{m-1}$ $(m > 0)$.

(iv) *For each* $a \in M$, $[S_a^m] \in \mathcal{M}$.

(v) *If a_1 and a_2 satisfy the same element of X^m then* $S_{a_1}^m \simeq S_{a_2}^m$.

Now since $B_2(T(M))$ is finite, there must be an N such that for $n \geqslant N$,

$$M \models \neg(\mathrm{E}x)(\mathrm{E}y)\big(C_n(x, y) \wedge \neg C_{n+1}(x, y)\big).$$

Consider $\mathbf{C}_N$; suppose that $S_{a_1}^N < S_{a_2}^N$ and that for no $b \in M$ we have $S_{a_1}^N < S_b^N < S_{a_2}^N$. Let S be a maximal discrete segment of $\mathbf{C}_N$; certainly S cannot be infinite, for otherwise the following infinite set of wffs are pairwise inequivalent in $B_2(T(M))$:

$$(v \geqslant 2)\ (\mathrm{E}x_1)(\mathrm{E}x_2)\dots(\mathrm{E}x_v)\Big[x = x_1 < x_2 < \dots < x_v = y \wedge \bigwedge_{1 \leqslant i < v} \neg C_N(x_i, x_{i+1}) \wedge$$
$$\wedge (w)\Big((x \leqslant w \leqslant y \Rightarrow \bigvee_{1 \leqslant i \leqslant v} \big(C_N(x_i, w) \wedge C_N(w, x_i)\big)\Big)\Big].$$

On the other hand if S is finite then at the next stage they will be combined. Hence we conclude that $\mathbf{C}_N$ has dense order type.

To complete the proof we need only show that the order type of $\mathbf{C}_N$ is 1, because together with (iv) this implies that $[M] \in \mathcal{M}$. But by (v) the splitting $\mathbf{C}_N$ has only finitely many distinct parts; hence, if $\mathbf{C}_N$ is not of order type 1, the lemma below gives a segment of $\mathbf{C}_N$ which would be combined into one part of $\mathbf{C}_{N+1}$. This is impossible by assumption.

LEMMA. *If an interval I of the rational line is partitioned into k sets $R_1, R_2, \ldots, R_k$, then there is a subinterval $I^* \subseteq I$ and a subset $\{i_1, i_2, \ldots, i_s\}$ of $\{1, 2, \ldots, k\}$ such that if $(x, y) \subseteq I^*$ then for each j, $1 \leqslant j \leqslant s$, $(x, y) \cap R_{i_j} \neq \emptyset$.*

Proof. By induction on k. There is nothing to prove for $k = 1$. So assume it is true for $k-1$. Let i_0 be such that for some $(a, b) \subseteq I$, $(a, b) \cap R_{i_0} = \emptyset$; if none such exist then I and $\{1, 2, \ldots, k\}$ satisfy the conclusion of the lemma. But now (a, b) is partitioned into $k-1$ sets so the induction hypothesis proves the result.

Note added in proof: H. Lauchli has shown independently that, for a linear ordering M, $[M] \in \mathfrak{M}$ if and only if $J(M)$ is $\aleph_0$-categorical and finitely axiomatizable. By the proof above, for a linear ordering, finite axiomatizability follows from $\aleph_0$-categority.

References

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