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EFFECTIVE MATCHMAKING (RECURSION THEORETIC ASPECTS OF A THEOREM OF PHILIP HALL)

By ALFRED B. MANASTER† *and* JOSEPH G. ROSENSTEIN‡

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Given a set B of boys and a set G of girls, we call a subset S of $B \times G$ a *society* and we say that b *knows* g when $\langle b, g \rangle \in S$. The marriage problem for the society S is said to be *solvable* if it is possible to marry, in the traditional one-to-one manner, each boy to a girl whom he knows. We are concerned here with the computable analogues of these notions. Thus a society is *recursive* if there exists an algorithm which, when presented with a boy b and a girl g, effectively determines whether b knows g. Similarly, the marriage problem for the society S is said to be *recursively solvable* if there exists a one-to-one algorithm which, when presented with a boy b, effectively marries him to a girl whom he knows. We first show that, even if (the marriage problem for) a recursive society is solvable, it need not be recursively solvable. We then consider several conditions on solvable recursive societies; for each we determine whether such a society must be recursively solvable and, if not, how computationally complex its solutions need be. We also discuss some sociological variations of the marriage problem and indicate how our results can be applied to them.

We have drawn upon ideas from two branches of mathematics—combinatorics and recursive function theory. The combinatorial motivation has its source in a famous theorem of Philip Hall ([**4**]) which implies that if there are only a finite number of boys, then the society S is solvable if and only if, for each natural number k, any k distinct boys know among them at least k different girls. Using a compactness argument one can show that this same condition is necessary and sufficient even if there are an infinite number of boys, so long as no boy knows infinitely many girls. (See either [**5**] for a combinatorial argument or [**1**], p. 47, for a proof based on the propositional calculus. This generalization was first proved by M. Hall ([**3**]). L. Mirsky's new book ([**10**]) contains an exhaustive

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account of the ramifications of P. Hall's theorem.) Recursive function theory, on the other hand, provides the tools with which one can measure the degree of effectiveness or computability of a function or a set whose existence is proved by combinatorial means. An important application of these techniques is the recent work of Jockusch ([**6**]) on the combinatorial theorem of Ramsey, which influenced the first author of the present paper. We feel that computable versions of combinatorial results can help contribute to our understanding and appreciation of the combinatorial arguments; we also hope that such analyses may eventually provide a precise way of measuring the complexity of various combinatorial results.

The paper is essentially self-contained. The reader who is not familiar with recursive function theory and mathematical logic will find all the information that he needs in the section below. He will also find a guide to that section so that he can tell which concepts and terminology he needs to understand different parts of the paper. The reader who is familiar with recursion theory can omit this section, for standard notation is used throughout the paper.

1. *Guide.* This section, containing background information about recursive function theory and mathematical logic, is divided into three parts. The first part 'Algorithms and partial recursive functions' contains all the material necessary for Theorems 1–4 of § 2 and Theorems 1*–4* of § 3. The second part 'Relative recursiveness' is needed for Theorems 5–8 of § 2 and Theorems 5*–8* of § 3. The third part 'The jump operator' is needed only for the remaining theorems.

Algorithms and partial recursive functions. In the opening paragraphs above, two algorithms are mentioned. The first, when presented with a boy b and a girl g, determines whether b knows g; the second, when presented with a boy b, finds a girl whom he should marry. In this paper the boys in a society will always be $\{B(i) \mid i \in N\}$ (where N is the set of natural numbers) and the girls will always be $\{G(j) \mid j \in N\}$. Thus the first algorithm can be viewed as determining a function $f\colon N \times N \to N$ such that $f(i,j) = 1$ if $B(i)$ knows $G(j)$ and $f(i,j) = 0$ otherwise; similarly, the second algorithm can be viewed as determining a function $h\colon N \to N$ such that $h(i) = j$ means that $B(i)$ should marry $G(j)$. We shall therefore assume that each algorithm we consider determines a function $g\colon N^k \to N$ for some k (where N^k is here the Cartesian product for k copies of N).

What is an algorithm? Intuitively, an algorithm involves a *finite* set of instructions, formulated in a finite language, which, given any argument, generates a computation which, after a *finite* number of operations, yields an answer.

There have been many attempts to make precise this intuitive idea of an algorithm. Most notable are those of Herbrand and Gödel, Kleene, Turing, Markov (see [**9**], chapter 5, or [**8**]), Church ([**2**]), and Post ([**13**]). Each such attempt leads to a class of functions, and in fact these classes turn out to be the same.

The fact that each attempt to formalize the notion of algorithm leads to the same class of functions resulted in Church's thesis (discussed in [**8**], §§ 62 and 70), which claims that the intuitive notion of algorithm has indeed been captured by the formal definitions referred to above. We shall assume Church's thesis, and, following Kleene, shall call the class of functions determined by algorithms the *class of general recursive functions*, or, more briefly, the *class of recursive functions*.

Unfortunately the class of general recursive functions is not the most natural class of functions to deal with. The reason for this is that given an arbitrary finite set of instructions and an argument, the computations generated may not yield an answer; this means that a given finite set of instructions may not be an algorithm. Even more seriously, if we assign a code number to each finite set of instructions, there is no algorithm which, when presented with an ordered pair $\langle x, y\rangle$ of natural numbers, will determine whether the set of instructions numbered x will, when presented with the argument y, yield an answer; this means that there is no effective way of telling whether a finite set of instructions is indeed an algorithm.

It is therefore more natural to consider finite sets of instructions, or partial algorithms, rather than algorithms themselves. Any finite set of instructions determines a function whose domain is some subset of N^k (for the appropriate k) or, more succinctly, any partial algorithm determines a partial function. Each of the attempts mentioned above to formalize the notion of algorithm extends naturally to a formalization of the notion of partial algorithm. Again the resulting classes of partial functions are identical; and, in this context, Church's thesis claims that the formal definitions capture the intuitive notions of 'partial algorithm', or 'finite set of instructions'. We shall use Kleene's term—*partial recursive function*—for a partial function determined by a partial algorithm.

Since the class of partial algorithms is denumerable, it is possible to assign to each finite set of instructions a code number; assuming that this has been done, we let φ_e denote the partial function determined by the finite set of instructions numbered e, and we write $\varphi_e(x) \simeq y$ if φ_e is defined at x and has value y. (Assuming that finite sets of instructions involving one argument are enumerated separately from those requiring two arguments, we are using φ_e ambiguously—but this should lead to no confusion.)

We shall write $\varphi_e^n(x) = y$ if, when presented with the argument x, the set of instructions with number e will compute the answer y in at most n operations or steps. The notion of 'in at most n steps' can be defined precisely in such a way that we may draw the following conclusions:

(i) if $\varphi_e^n(x) = y$ then, for all $m > n$, $\varphi_e^m(x) = y$ and thus if $\varphi_e^m(x)$ is undefined and $m > n$ then $\varphi_e^n(x)$ is also undefined;

(ii) $\varphi_e(x) \simeq y$ if and only if there is an n such that $\varphi_e^n(x) = y$;

(iii) if $\varphi_e^n(x) = y$ then $n > e$, $n > x$, and $n > y$.

We stress that the enumeration of all finite sets of instructions, in the paragraphs above, is not a haphazard one. On the contrary, this enumeration can be executed by an algorithm—that is, there is a partial recursive function $\varphi(x, y)$ such that φ enumerates all partial recursive functions of one variable; more precisely, for each e and y, $\varphi(e, y)$ is defined if and only if $\varphi_e(y)$ is defined and, in case both are defined, they are equal. (This condition is expressed by the 'equation' $\varphi(e, y) \simeq \varphi_e(y)$.) Thus the set of instructions for the partial recursive function $\varphi(x, y)$ contains 'within it' all finite sets of instructions for functions of one variable—since if it is fed the ordered pair $\langle e, y\rangle$ it in effect turns to the algorithm for φ_e and feeds into that the argument y. The existence of a so-called universal partial recursive function, sometimes referred to as the enumeration theorem, will be essential in the sequel, although we shall usually use it tacitly.

In the last few paragraphs we have discussed only the partial recursive functions of one variable. The same discussion, however, could have been presented for the partial recursive functions of k variables. Thus the enumeration theorem, for partial recursive functions of k variables, states that there is a partial recursive function $\varphi(x, y_1, y_2, \ldots, y_k)$ of $k+1$ variables such that for each e and each $y_1, y_2, \ldots, y_k$ the equation

$$\varphi(e, y_1, y_2, \ldots, y_k) \simeq \varphi_e(y_1, y_2, \ldots, y_k)$$

holds.

A relation $R \subseteq N^k$ is said to be *recursive* if its characteristic function X_R is a general recursive function (where X_R is defined by $X_R(x) = 1$ if $x \in R$ and $X_R(x) = 0$ if $x \notin R$). Intuitively a relation is recursive if there is an algorithm which will determine whether or not each particular element of N^k is in the relation.

The remaining material in this part of this section ('Algorithms and partial recursive functions') is not needed for understanding the statements or the proofs of Theorems 1–4 and Theorems 1*–4*. It will, however, be needed to formalize the proofs; this formalization will play a more important role in some of the later theorems and is therefore included.

We mention here a number of recursive functions and recursive relations and ways to get new recursive functions, sets, and relations from old ones. The usual arithmetic operations $f(x,y) = x+y$, $f(x,y) = x.y$, $f(x,y) = x^y$ are all recursive, as is $f(x,y) = x \dot{-} y = \max(x-y, 0)$. If f is a recursive function then $g(n) = \sum_{x<n} f(x)$ is also recursive. The function $f(x) = P_x$ is recursive, where P_x is the xth prime, as is the function $f(s,i) = (s)_i$, the power of the prime P_i in the unique factorization of s. If f is a recursive function then so is $h(n) = \prod_{x<n} P_x^{f(x)}$; since $(h(n))_x = f(x)$ for each $x < n$, $h(n)$ can be thought of as an encoding of the first n values of the function f. The function h is thus called the *course-of-values function* for f and is usually denoted by $\bar{f}$—so that $(\bar{f}(n))_x = f(x)$ for $x < n$ and $(\bar{f}(n))_x = 0$ for $x \geqslant n$. Any number s can be thought of as encoding a finite sequence of numbers; thus if $s = 2^{s_0}3^{s_1}\ldots P_{n-1}{}^{s_{n-1}}$, where $s_{n-1} > 0$, then s can be thought of as encoding $\langle s_0, s_1, \ldots, s_{n-1}\rangle$. We define the function $g(s) = \mathrm{lh}\, s$, the length of s, to be n if s is as above. Thus $\mathrm{lh}(\bar{f}(n)) \leqslant n$. This function is also recursive. The set of recursive functions is closed under composition and under various 'recursive' definitions—for example, if h is recursive and the function f satisfies the identities $f(n) = h(\bar{f}(n))$, then f is also recursive, for intuitively the given algorithm for h can be converted into an algorithm which will effectively compute new values for f from old ones. The set of recursive relations is closed under union, intersection, and complementation. If one thinks of a relation as a 'predicate' rather than as a set of k-tuples, then what we have just said is that the recursive predicates are closed under disjunction, conjunction, and negation. Also if $R(x,y)$ is a recursive predicate then the predicate $(\exists x)_{x<n} R(x,y)$—'there is an $x < n$ such that $R(x,y)$'—is a recursive relation of n and y, since it is easy to build an algorithm for this predicate from a given one for R. On the other hand, the predicate $(\exists x)\, R(x,y)$ need not be recursive since the natural algorithm to build will end up searching for an x and, if it fails to find one, will keep searching, even if no x exists. Similarly $(\forall x)_{x<n} R(x,y)$—'for all $x < n$, $R(x,y)$'—is a recursive predicate if R is, but $(\forall x)\, R(x,y)$ need not be. Thus recursive relations are not closed under projection. If $R(x,y)$ is a recursive relation then the function $f(y) = \mu x\, R(x,y)$—'the least x such that $R(x,y)$'—need not be recursive, since for a fixed y, there may be no such x. But, if R is a recursive relation and $(\forall y)\,(\exists x)\, R(x,y)$ holds then the function $f(y) = \mu x\, R(x,y)$ is recursive and it is clear how to build an algorithm for f from an algorithm for R. Finally, if f is a recursive function and g is any function which agrees with f in all but a finite number of places, then g too is recursive, for we can construct an algorithm which at that finite set of arguments gives the answers appropriate for g and otherwise will act just like the original algorithm for f. Further details about recursive functions and relations can be found in [**14**] or in [**9**], pp. 120–30, but this sketch includes the information about them needed for this paper; that is, anything we claim to be recursive is recursive by an argument which is related to what appears above.

Relative recursiveness. Let A be a fixed set of natural numbers. Suppose that we modify the notion of partial algorithm so that, in the course of a computation, questions of the form 'Is $n \in A$?' may be asked for any number n. When such a question is asked, the computation may continue in one way if an answer of 'Yes' is received and another way if 'No' is the answer. The partial function obtained from such a partial algorithm is said to be *partial recursive relative to* A, or, simply, *partial recursive in* A. Intuitively a partial function is partial recursive in A if there is a finite set of instructions such that, if the computation it describes is performed using answers supplied by an 'oracle' which knows (the characteristic function of) A, then the given partial function is obtained. For each set A there is an enumeration (by a 2-place function partial recursive in A) of all the 1-place functions partial recursive in A. The one determined by the finite set of instructions numbered e is denoted φ_e^A.

Instead of using an oracle which knows (the characteristic function of) the set A, we could use an oracle which knows a given (arbitrary) function f and ask questions like 'What is $f(n)$?' in the course of a computation. In this case the partial function determined by the finite set of instructions numbered e is denoted φ_e^f. If a partial function is partial recursive in f and is everywhere defined then it is said to be recursive in f.

We have thus defined a relation on the set of all functions. If one identifies a set with its characteristic function, we can give meaning to the phrases 'A is recursive in B', 'A is recursive in f', and 'f is recursive in A'. The relation 'X is recursive in Y' is both transitive and reflexive, and is denoted by $X \leqslant Y$. We say that X and Y have the same *degree of unsolvability* if $X \leqslant Y$ and $Y \leqslant X$; thus a 'degree of unsolvability' can be viewed as an equivalence class. The degrees of unsolvability are partially ordered by the relation $\leqslant$. The lowest degree of unsolvability is the equivalence class consisting of the recursive sets, since a recursive set is recursive in any other set; this degree is denoted by $\mathbf{0}$.

A set of natural numbers is said to be recursively enumerable if it is the range of a partial recursive function—thus for example, the set of primes is recursively enumerable since it is the range of the partial recursive function P described above. Every recursive set is recursively enumerable but there are recursively enumerable sets which are not recursive. Thus there are recursively enumerable sets whose degree of unsolvability exceeds **0**. The degrees of unsolvability of recursively enumerable sets, however, cannot be 'arbitrarily large'. To verify this, one first observes that, since there are a countable number of partial recursive functions, there are a countable number of recursively enumerable sets. One then defines a set A, whose members are all prime powers, by stipulating that $P_n^x \in A$ if an only if x is an element of the nth recursively enumerable set in some fixed enumeration of the recursively enumerable sets. It is clear that each recursively enumerable set is in fact recursive in A and, thus, that the degree of A is an upper bound for the degrees of all recursively enumerable sets; that it is not an upper bound for the degrees of all sets is a consequence of the fact that only a countable number of sets are recursive in A. This argument can be refined by first showing that there is a *recursive* enumeration of the recursively enumerable sets; the following conclusion is then obtained.

There is a recursively enumerable set such that every recursively enumerable set is recursive in it; the degree of such a set is denoted by $\mathbf{0}'$. An example of a recursively enumerable set of degree $\mathbf{0}'$ is the set K of all numbers e such that $\varphi_e(e)$ is defined. This set K is the range of the partial recursive function ψ computed according to the instructions 'Try to compute $\varphi_e(e)$; if you successfully complete a computation of $\varphi_e(e)$, then put $\psi(e)$ equal to e.'

The class of recursively enumerable sets can be characterized as the class of projections of recursive relations. That is, A is recursively enumerable if and only if there is a recursive relation R such that for every x, $x \in A$ if and only if $(\exists\, y)\, R(x,y)$. To verify this, we first assume that A is the range of the partial recursive function ψ and let $R(x,y)$ be the relation 'the computation of $\psi((y)_0)$ is completed in at most $(y)_1$ steps and the result of the computation is x'; clearly $x \in A$ if and only if $(\exists\, y)\, R(x,y)$. On the other hand, given a recursive relation R, let ψ be the partial recursive function determined by the following instructions: 'To find $\psi(x)$, determine whether $R(x,0)$ holds, whether $R(x,1)$ holds, et cetera, and as soon as a number y is found for which $R(x,y)$ holds, put $\psi(x)$ equal to x.'

More generally a relation $P \subseteq N^k$ is recursively enumerable if and only if there is a recursive relation $R \subseteq N^{k+1}$ such that for every $\langle x_1, x_2, \ldots, x_k\rangle$,

$$\langle x_1, x_2, \ldots, x_k\rangle \in P \quad \text{if and only if } (\exists\, y)\, R(x_1, x_2, \ldots, x_k, y).$$

Now any recursively enumerable set is recursive in $\mathbf{0}'$, so that any projection of a recursive relation is recursive in $\mathbf{0}'$. Consider a set A defined by $x \in A$ if and only if $(\forall\, y)\, R(x,y)$, where R is a recursive relation.

The complement $\bar{A}$ of A thus satisfies $x \in \bar{A}$ if and only if $(\exists y) \sim R(x,y)$ and $\sim R$ is a recursive relation so that $\bar{A}$ is recursively enumerable. Since A and $\bar{A}$ have the same degree of unsolvability, any set (or relation) defined by one quantification of a recursive relation is recursive in $\mathbf{0}'$.

If $\mathbf{d}$ is any degree $\leqslant \mathbf{0}'$ and D is a set of degree $\mathbf{d}$ then, since X_D is recursive in $\mathbf{0}'$, there is a number c such that $X_D = \varphi_c^K$. There are two ways in which one could attempt to compute X_D. One is simply to use an oracle for K, as described earlier in this section. Another is to use the fact that K is recursively enumerable and, rather than assume K to be given, try to construct it in the course of the computation. Thus, for example, one can define K^n to be $\{e \mid \varphi_e(e)$ is defined in at most n steps$\}$ so that K^n is approximately what K appears to be after n steps in its enumeration. One could then, to determine whether $a \in D$, find for each n the set K^n and attempt n steps of the computation of $\varphi_c^{K^n}(a)$. The value thus obtained, if it exists, will be denoted $\varphi_c^{K,n}(a)$. Because of the description above this can be thought of as a partial recursive function of the three variables c, n, and a. As n increases the sets K^n will change in appearance so that $\varphi_c^{K,n}(a)$ may take on many different values for fixed a and c as n increases. If, however, $\varphi_c^K(a) = b$ then for sufficiently large n all properties of K used oracularly in the computation will be shared by K^n and thus, for all $m \geqslant n$, $\varphi_c^{K,m}(a) = b$. We shall use this description of φ_c^K in Theorems 7 and 8.

The jump operator. The discussion of the preceding section which led from $\mathbf{0}$ to $\mathbf{0}'$ can be repeated in a more general way to go from an arbitrary degree $\mathbf{d}$ to its jump $\mathbf{d}'$. Thus, given a degree $\mathbf{d}$ we say that a set A is recursively enumerable in $\mathbf{d}$ if there is a function which is partial recursive in $\mathbf{d}$ (i.e. in any set of degree $\mathbf{d}$) whose range is A. There are sets which are recursively enumerable in $\mathbf{d}$ which are not recursive in $\mathbf{d}$. In particular, if D is any set of degree $\mathbf{d}$ then the set K^D described by $e \in K^D$ if and only if $\varphi_e^D(e)$ is defined is recursively enumerable in $\mathbf{d}$ and every set recursively enumerable in $\mathbf{d}$ is recursive in it. Its degree does not depend on the choice of D so we may define $\mathbf{d}'$ to be the degree of K^D.

If A is recursively enumerable in $\mathbf{d}$ then there is a predicate $R(x,y)$ recursive in $\mathbf{d}$ such that $x \in A$ if and only if $(\exists y) R(x,y)$. Conversely, if $R(x,y)$ is recursive in $\mathbf{d}$ then the set $\{x \mid (\exists y) R(x,y)\}$ is recursively enumerable in $\mathbf{d}$ and therefore recursive in $\mathbf{d}'$. Similarly any relation $P \subseteq N^k$ for which there is a relation $R \subseteq N^{k+1}$ recursive in $\mathbf{d}$ such that $\langle x_1, \ldots, x_k \rangle \in P$ if and only if $(\exists y) R(x_1, \ldots, x_k, y)$ is also recursive in $\mathbf{d}'$. Also if there is a relation $R \subseteq N^{k+1}$ recursive in $\mathbf{d}$ such that $\langle x_1, \ldots, x_k \rangle \in P$ if and only if $(\forall y) R(x_1, \ldots, x_k, y)$, then $\langle x_1, \ldots, x_k \rangle \in N^k \backslash P$ if and only if

$(\exists y) \sim R(x_1, \ldots, x_k, y)$ so that $N^k \backslash P$ is recursive in $\mathbf{d}'$ and therefore so is P.

Now assume that there is a recursive relation $R(x, x_2, \ldots, x_k, y, z)$ such that $\langle x_1, x_2, \ldots, x_k \rangle \in P$ if and only if

$$(\forall z)(\exists y) R(x_1, \ldots, x_k, y, z).$$

Then the predicate $(\exists y) R(x_1, \ldots, x_k, y, z)$ is recursive in $\mathbf{0}'$ and so P is recursive in $\mathbf{0}'' = (\mathbf{0}')'$.

2. *Definitions.* (1) An even natural number is called *a boy*; an odd natural number is called *a girl*. We use $B(n)$ for $2n$ and $G(n)$ for $2n+1$ and say that $B(n)$ is the nth *boy* and that $G(n)$ is the nth *girl*.

(2) If R is a binary relation on N (the set of natural numbers), we shall say that 'x knows$_R$ y' instead of '$\langle x, y \rangle \in R$'. If there is no danger of confusion we shall say simply '*x knows y*'.

(3) A *society* is a binary relation R on N such that (i) R is symmetric, (ii) if x knows y then x and y are of different sexes, (iii) each person knows just a finite number of people, and (iv) each person knows some other person. If the relation R satisfies just (i), (ii), and (iii), then it is called a *partial society*.

(4) If R is a partial society we partition the field of R into equivalence classes, called *communities of R*, by stipulating that 'x and y are in the same community' if there is a finite sequence $x_1, x_2, \ldots, x_n$ such that x_1 is x, x_n is y, and x_i knows x_{i+1} for $1 \leqslant i < n$. If C is a community of R, and if there is no possibility of confusion, we shall speak of 'the community C' whenever we have in mind 'the partial society $(C \times C) \cap R$'.

(5) If R is a partial society and f is a one-to-one function whose domain is $\{n \mid B(n)$ is in the field of $R\}$, then f is called a *solution to the marriage problem of R* or, briefly, a *solution of R*, if $B(n)$ knows $G(f(n))$ for every n in the domain of f.

(6) A partial society R is said to be *solvable* if there is a solution to the marriage problem of R.

Marshall Hall Jr's extension of the Philip Hall theorem, mentioned in the opening paragraphs, implies that a partial society R is solvable if and only if every n boys know among them at least n girls.

It is evident that a partial society is solvable if and only if each of its communities is solvable; it has a unique solution if and only if each of its communities has a unique solution.

(7) A society S is said to be *recursively solvable* if there is a general recursive function f which is a solution to the marriage problem of S.

The alert reader will have noticed that these definitions embody various sociological biases. For example, a solution to the marriage problem of a society S assigns a partner to each boy but may leave certain girls unattached. This sexist attitude is not essential, as will become clear in § 3, where we shall deal with the symmetric marriage problem. In § 4 we shall discuss briefly other sociological variations of the marriage problem.

Conventions. The proofs of Theorems 1, 3, 7, and 8 have certain similarities which we mention here. In each case we shall define a recursive society S by stages—i.e. at stage n, for each $n > 0$, we shall define a partial society S_n so that $S = \bigcup_{n>0} S_n$ has the desired properties. Instead of saying 'put $\langle x, y\rangle$ into S_n' during stage n, we shall say 'introduce x to y' during stage n. A person is called a *stranger* at a given point in the construction if at that point he has not yet been introduced to anyone. At the beginning of each stage of the construction, there are numbers a and b such that the first a boys and b girls are not strangers whereas the remaining boys and girls are; we shall reserve the letters a and b for this purpose, so that $B(a)$ and $G(b)$ will always denote the first male and female strangers, respectively.

The community to which the boy $B(i)$ belongs at the beginning of stage n will be denoted $C_n(i)$. (Note that $C_n(i)$ is a community of the partial society S_{n-1}.) We shall call the community $C_n(i)$ *stable* if $C_m(i) = C_n(i)$ for all $m \geqslant n$. In the course of the construction no two non-strangers will ever be introduced; in particular, if $C_n(i)$ is stable this implies that no member of $C_n(i)$ will ever meet someone new. Hence if $C_n(i)$ is stable and $B(p)$ and $G(q)$ are in $C_n(i)$, then if $B(p)$ cannot marry $G(q)$ in $C_n(i)$ (i.e. if there is no solution to the marriage problem of $C_n(i)$ in which $B(p)$ marries $G(q)$), he cannot marry her in S, and if $B(p)$ must marry $G(q)$ in $C_n(i)$ he must marry her in S.

THEOREM 1. *There is a recursive society S which is solvable but not recursively solvable.*

Proof. We precede the actual construction of S with an intuitive discussion of how it works. In the course of the construction, we must guarantee that no general recursive function is a solution to the marriage problem of S; however, it is more natural to construct S so that for each e the partial recursive function φ_e cannot be a solution to the marriage problem of S. To do this it is sufficient to make certain that, for each e, there is some boy who cannot marry the girl whom the eth partial recursive function would assign to him. Therefore, for each e we set aside a boy

$B(r(e))$ and give him a mate, with the understanding that if we later discover that she is actually the girl $G(\varphi_e(r(e)))$ whom the eth partial recursive function would have him marry, then we will take steps to prevent the marriage. The action we take in such a case is to introduce a new boy to $G(\varphi_e(r(e)))$ and a new girl to $B(r(e))$. If we ensure that the resulting community is stable then in any solution to the marriage problem for S the new boy will win the hand of $G(\varphi_e(r(e)))$ and therefore $B(r(e))$ will have to marry the new girl.

The only difficulty with this plan is that when we try to compute $\varphi_e(r(e))$ to see whom φ_e wants $B(r(e))$ to marry, no answer may be forthcoming. We cannot keep trying to compute $\varphi_e(r(e))$ for ever, since it is possible that the computation will never terminate and since it is necessary for the construction to continue. So we agree that if at stage n we are concentrating on φ_e, then we perform at most n steps of the computation of $\varphi_e(r(e))$, and, if we still get no answer, then we abandon the computation and proceed to stage $n+1$. But in that case there must be some provision made for resuming the computation of $\varphi_e(r(e))$ later on, because it is quite possible that with additional time the computation of $\varphi_e(r(e))$ would terminate. We therefore arrange matters so that for each e there are infinitely many stages of the construction when we concentrate on φ_e—specifically, we concentrate on φ_e at each of the infinitely many stages n when $n = 2^e.q$ for some odd number q. We make critical use of the fact that if $\varphi_e(r(e))$ is defined, then it is defined in finitely many steps and hence by some stage n; for using this fact we know that if some action is necessary, then the construction will afford us an opportunity to take that action.

With this description in mind we proceed to define simultaneously the society S and a one-to-one recursive function r; at the end of stage n, $r(e)$ will be defined for all $e < n$.

Stage n. Introduce $B(a)$ to $G(b)$ and let $r(n-1) = a$. Suppose that $n = 2^e.q$, where q is odd. If $\varphi_e^n(r(e))$ is defined (i.e. if $\varphi_e(r(e))$ is defined in at most n steps), if $B(r(e))$ knows $G(\varphi_e^n(r(e)))$, and if these two comprise the community $C_n(r(e))$, then we introduce $B(r(e))$ to $G(b+1)$ and $G(\varphi_e^n(r(e)))$ to $B(a+1)$. Otherwise we proceed directly to stage $n+1$.

This completes the definition of the construction. We must show that

(i) S is recursive,

(ii) S is solvable,

(iii) S is not recursively solvable.

As to (i), we note first that in any introduction at least one of the partners being introduced is at that time a stranger. We note also that $B(n)$ and $G(n)$ are no longer strangers by the end of stage $n+1$. Hence to decide whether or not $B(n)$ knows $G(m)$ we need only to reconstruct effectively (using the enumeration theorem) the first $n+m$ stages of the construction above and see whether or not they have been introduced by that stage. Thus the recursiveness of S is guaranteed by the fact that we never introduce two non-strangers in the course of the construction. Note that in claiming to have proved that S is recursive, we are making use of Church's thesis which says that the algorithm described here can be formalized. If we did that it would become very evident that critical use is being made here of the enumeration theorem as was claimed earlier.

As to (ii), we need only remark that for each e and each $n > e$ the community $C_n(r(e))$ is solvable, and furthermore that for sufficiently large n, $C_n(r(e))$ is stable.

As to (iii), let f be any recursive function, say f is φ_e. Since f is recursive, $\varphi_e(r(e))$ is defined. Choose n sufficiently large so that $r(e)$ is defined before stage n, $\varphi_e^n(r(e))$ is defined (and hence equal to $\varphi_e(r(e))$), and so that $n = 2^e.q$, where q is odd. At the end of stage n it will not be possible for $B(r(e))$ to marry $G(\varphi_e^n(r(e)))$ and, since $C_{n+1}(r(e))$ is stable, nothing done later on can ever bring them together.

The society constructed in the proof of Theorem 1 has a unique solution to its marriage problem. Thus the condition that the society have a unique solution is not sufficient to guarantee that a recursive society be recursively solvable. The following condition, in conjunction with the condition above, will be sufficient.

A recursive society S is said to be *highly recursive* if there is a recursive function h such that, for each i, the number of girls whom $B(i)$ knows is exactly $h(i)$.

To see the effect of this condition, let us imagine a matchmaker in a recursive society. If he were asked to find a partner for the boy $B(i)$ he might start by listing out all the girls whom $B(i)$ knows. He would do this by consecutively asking whether $B(i)$ knows $G(0)$, whether $B(i)$ knows $G(1)$, et cetera. Since $B(i)$ knows only a finite number of girls the matchmaker would eventually complete the list of girls that $B(i)$ knows. Unfortunately he may not know when he has completed the list. If he stops at any $G(j)$ he runs the risk that perhaps $B(i)$ knows $G(j+1)$, and in fact should marry her; if he keeps going forever, he will never get to his job of marrying off $B(i)$.

In a highly recursive society, the matchmaker's task is much easier; for he can first determine effectively how many girls $B(i)$ knows, and once he has found that many girls, he knows that his list is complete and can then get on with his job.

The intuitive description above of the difference between a recursive society and a highly recursive society is reflected in the proof of the theorem below.

THEOREM 2. *If S is a highly recursive society and the marriage problem for S has a unique solution, then that solution is recursive.*

Proof. We shall describe an algorithm which the matchmaker can use to find the solution to the marriage problem for S. The fact that the algorithm works is a consequence of the combinatorial fact that

(*) if R is a partial society which has a unique solution to its marriage problem then some boy in the society must know only one girl.

This statement is a consequence of the lemma which is proved below.

The matchmaker begins by computing $h(0)$, $h(1)$, $h(2)$, ... until he finds a boy who knows only one girl. That this search will be successfully completed follows from (*) and the assumption that S does in fact have a unique solution to its marriage problem. Let $B(i_0)$ be the boy that he finds who knows only one girl; he now proceeds to determine whether $B(i_0)$ knows $G(0)$, whether $B(i_0)$ knows $G(1), \ldots$ until he finds the unique girl whom $B(i_0)$ knows. Let $G(j_0)$ be this girl. It is clear that in the unique solution to the marriage problem of S, the boy $B(i_0)$ must actually marry $G(j_0)$.

Now let R_1 be the partial society obtained from S by deleting $B(i_0)$ and $G(j_0)$. Since R_1 must have a unique solution to its marriage problem, the matchmaker realizes (using the lemma) that some boy in R_1 must know only one girl in R_1. His task is to find such a boy. He could of course continue computing $h(i_0+1)$, $h(i_0+2)$, ... until he finds some i for which $h(i) = 1$, but this search may never end, for it could happen that the boy he seeks in fact knows two girls one of whom is $G(j_0)$. To take this possibility into account he must go back to the beginning—considering $h(0)$, $h(1)$, $h(2)$, ... until he finds a boy who knows exactly one girl of R_1. With each i, he computes $h(i)$; if $h(i) > 2$ he goes on to $i+1$, and if $h(i) = 2$ he determines whether $B(i)$ knows $G(j_0)$ and if not he goes on to $i+1$; but if $h(i) = 2$ and $B(i)$ knows $G(j_0)$ or if $h(i) = 1$ and $i \neq i_0$, then he stops and identifies $B(i)$ as $B(i_1)$ and the unique girl of R_1 he knows as $G(j_1)$. It is clear that in the unique solution to the marriage problem of S, the boy $B(i_1)$ must actually marry $G(j_1)$.

The matchmaker proceeds inductively. Assume that after a certain time he has identified n boys $B(i_0)$, $B(i_1)$, ..., $B(i_{n-1})$ and n girls $G(j_0)$, $G(j_1)$, ..., $G(j_{n-1})$ whom they must marry in the unique solution to the marriage problem of S. Let R_n be the partial society obtained from S

by deleting these n couples; then R_n has a unique solution to its marriage problem, hence by (*) there is a boy in R_n who knows exactly one girl of R_n. The matchmaker wants to find such a boy. He calculates consecutively $h(0)$, $h(1)$, $h(2)$, ... For each i different from $i_0, i_1, \ldots, i_{n-1}$ for which $h(i) \leqslant n+1$ he determines which of $G(j_0), G(j_1), \ldots, G(j_{n-1})$ the boy $B(i)$ knows and subtracts their number from $h(i)$. If the result is 1 he has his man; otherwise he goes on to the next boy. Since (*) holds he knows that he will be able to find a boy $B(i_n)$ and a girl $G(j_n)$ whom he must marry in the unique solution to the marriage problem of S.

This completes the description of the algorithm. We must now show that it works. That is, suppose a boy $B(t)$ comes to the matchmaker seeking his wife; can the matchmaker find her for him?

The matchmaker sets the algorithm into progress from the beginning and pairs off $B(i_0)$ with $G(j_0)$, $B(i_1)$ with $G(j_1)$, ... He actually finds a partner for $B(t)$ if and only if $B(t)$ is $B(i_r)$ for some r. So we must show that this is indeed the case.

Suppose then that $B(t)$ is none of the boys $B(i_0), B(i_1), \ldots$; hence in the unique solution to the marriage problem of S he marries some girl other than $G(j_0)$, $G(j_1)$, ... If he knows exactly one girl other than $G(j_0)$, $G(j_1)$, ..., then, since he knows only a finite number of girls altogether, there is a k such that he knows exactly one girl other than $G(j_0), G(j_1), \ldots, G(j_k)$. But if that is the case, then he would be one of the boys $B(i_{k+1}), B(i_{k+2}), \ldots, B(i_{k+t})$. Thus each $B(t)$ who is not enumerated in the sequence $B(i_0), B(i_1), \ldots$ knows at least two girls not enumerated in the sequence $G(j_0), G(j_1), \ldots$. Thus the partial society $\bigcap_{n=1}^{\infty} R_n$, if it contains a boy, cannot be uniquely solvable because of (*). But it is uniquely solvable, hence it can contain no boy; therefore $B(t)$ must be one of the boys in the sequence $B(i_0), B(i_1)$, ... Hence by following this algorithm the matchmaker will effectively find a mate for any boy. Thus an application of Church's thesis guarantees that the solution is indeed a general recursive function.

For the reader who feels he would like to see the solution to the marriage problem of S defined explicitly (and for use in a later theorem) we present the following formal description.

We define a recursive predicate $A(s,t,z,n)$ which says that 's and t represent sequences of length n, $B(z)$ is a boy different from $B((s)_0)$, $B((s)_1), \ldots, B((s)_{n-1})$, and $B(z)$ knows exactly one girl besides $G((t)_0)$, $G((t)_1), \ldots, G((t)_{n-1})$.' Formally $A(s,t,z,n)$ is

$$(\mathrm{lh}\, s = \mathrm{lh}\, t = n) \wedge \bigwedge_{i<n} z \neq (s)_i \wedge \bigvee_{i=1}^{n+1} \left[h(z) = i \wedge \left(\sum_{j<n} X_R(B(z), G((t)_j)) \right) = i \dot{-} 1 \right].$$

(Here X_R is the characteristic function of the relation R, $\bigwedge_{i<n}$ means 'the conjunction over all $i < n$', and $\bigvee_{i=1}^{n+1}$ means 'the disjunction over all i, $1 \leqslant i \leqslant n+1$'.)

Define simultaneously two recursive functions

$$i(n) = \mu z\, A(\tilde{\imath}(n), \tilde{\jmath}(n), z, n)$$

$$j(n) = \mu\omega \left[R(B(i(n)), G(\omega)) \wedge \bigwedge_{r<n} \omega \neq (\tilde{\jmath}(n)_r) \right]$$

by a simultaneous course-of-values recursion.

Then $B(i(n))$ and $G(j(n))$ are respectively $B(i_n)$ and $G(j_n)$ found above.

Define $\pi(t) = \mu n\,(t = i(n))$. Then π is recursive and $B(t)$ is $B(i(\pi(t)))$ for each t. Finally, we define a recursive function γ by $\gamma(t) = j(\pi(t))$.

Then in the unique solution to the marriage problem of S the boy $B(t)$ marries $G(\gamma(t))$ for every t. This completes the proof.

Before we prove the lemma we present a corollary to the theorem just proved.

Corollary. *Let S be a highly recursive society whose marriage problem has a finite number of solutions. Then these solutions are all recursive.*

Proof. Let f be a solution to the marriage problem of S and let $f_1, f_2, \ldots, f_K$ be the remaining solutions. Let $B(r_1)$, $B(r_2)$, $\ldots$, $B(r_K)$ be such that $f(r_1) \neq f_1(r_1)$, $f(r_2) \neq f_2(r_2)$, $\ldots$, $f(r_K) \neq f_K(r_K)$. Let S^* be the society which differs from S only in that $B(r_t)$ knows just $G(f(r_t))$ for $1 \leqslant t \leqslant K$. Then S^* is a highly recursive society, since it is finitely different from S. Furthermore, S^* has a unique solution to its marriage problem, namely f. Hence f is recursive.

We now prove the lemma. That it implies the statement (*) used in the proof of the theorem is obvious.

Lemma. *Let R be a partial society in which each boy knows at least two girls. If R is solvable, then the marriage problem of R has at least two solutions.*

Proof. Let f be a solution to the marriage problem of R. Let $B(i_1)$ be any boy in R and let $f(i_1) = j_1$ so that the solution f marries $B(i_1)$ to $G(j_1)$. Let $G(j_2)$ be a different girl whom $B(i_1)$ knows. If the solution f leaves $G(j_2)$ unmarried, then we can define a new solution f^* which marries $B(i_1)$ to $G(j_2)$ and each other boy $B(i)$ to $G(f(i))$. So we may assume that $j_2 = f(i_2)$ so that f marries $B(i_2)$ to $G(j_2)$. Proceeding inductively we may assume that we have defined K different boys $B(i_1)$, $B(i_2)$, $\ldots$, $B(i_K)$ and K girls $G(j_1)$, $G(j_2)$, $\ldots$, $G(j_K)$ such that f marries $B(i_t)$ to $G(j_t)$ for $1 \leqslant t \leqslant K$

and such that $B(i_t)$ knows $G(j_{t+1})$ for $1 \leqslant t < K$. By assumption $B(i_K)$ knows some girl other than the one he marries. There are now three cases. If $B(i_K)$ knows $G(j_t)$ for some $t < K$ then we can define a new solution f^* which marries $B(i_K)$ to $G(j_t)$, $B(i_t)$ to $G(j_{t+1})$, $B(i_{t+1})$ to $G(j_{t+2})$, ..., $B(i_{K-1})$ to $G(j_K)$ and each other boy $B(i)$ to $G(f(i))$. If $B(i_K)$ knows some girl other than $G(j_1)$, ..., $G(j_K)$—call her $G(j_{K+1})$—and she remains unmarried in the solution f, then we can define a new solution f^* which marries $B(i_K)$ to $G(j_{K+1})$ and each other boy $B(i)$ to $G(f(i))$. Finally, if $G(j_{K+1})$ is married in the solution f then her partner is different from $B(i_1)$, ..., $B(i_K)$—call him $B(i_{K+1})$—and we can proceed to the next step of the induction.

There are thus two possibilities. Either along the way we find a second solution or the inductive process continues without end. That is, we define a sequence $(B(i_t)\,|\,t \geqslant 1)$ of boys and a sequence $(G(j_t)\,|\,t \geqslant 1)$ of girls such that f marries $B(i_t)$ to $G(j_t)$ for all $t \geqslant 1$ and such that $B(i_t)$ knows $G(j_{t+1})$ for all t. But if this is so, we can define a new solution f^* which marries $B(i_t)$ to $G(j_{t+1})$ for all $t \geqslant 0$ and which marries any other boy $B(i)$ to $G(f(i))$.

It is reasonable to suppose, on the basis of the evidence of Theorem 2, that any highly recursive society has a recursive solution. We now show that this supposition would be incorrect.

Definition: A society in which every person knows exactly k other people is called a *k-society*. It is clear that any recursive k-society is highly recursive.

Theorem 3. *There exists a recursive 2-society S which is solvable but not recursively solvable.*

Proof. We precede the actual construction of S with an intuitive discussion of how it works. As in the proof of Theorem 1 we must guarantee that, for each number e, φ_e cannot be a solution to the marriage problem of S. In that proof we set aside, for each e, a boy $B(r(e))$ and succeeded in arranging matters so that if $\varphi_e(r(e))$ were defined than $B(r(e))$ could not marry $G(\varphi_e(r(e)))$. Here we are operating under the additional constraint that each boy must know exactly two girls, so the society S cannot have communities like those in the society constructed in Theorem 1.

We will set aside, for each e, two boys $B(r_0(e))$ and $B(r_1(e))$ and guarantee that in any solution to the marriage problem for the society at least one of them will be unable to marry the girl assigned him by φ_e. If in the course of the construction there never appears to be any danger that φ_e

may be a solution then the communities of $B(r_0(e))$ and $B(r_1(e))$ remain distinct and keep expanding so that in S the community of $B(r_i(e))$ will look like Fig. 1 (where one person knows another if and only if there is a line joining them), so that each person indeed knows exactly two other people.

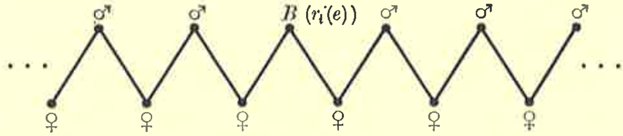

FIG. 1

If on the other hand, it does appear to be possible for φ_e to be a solution to the marriage problem of S—that is, if $\varphi_e(r_0(e))$ and $\varphi_e(r_1(e))$ are at some point defined and if $B(r_i(e))$ knows $G(\varphi_e(r_i(e)))$ for $i = 1, 2$ then we will take the communities that the two boys are in at that stage of the construction and splice them together in such a way that in no solution to the marriage problem of S can both of them marry the partners which φ_e has destined for them. (The precise definition of this 'splicing' necessitates the notational complexity found below.)

We proceed now to define simultaneously the society S and two one-to-one recursive functions r_0 and r_1, whose ranges are disjoint. At the end of stage n, both $r_0(e)$ and $r_1(e)$ will be defined for all $e < n$.

We assume as part of the inductive hypothesis that at the beginning of stage n each person who is not a stranger is in a community $C_n(r_t(e))$ for some $e < n-1$ and for some $t < 2$. We further assume that for each such community there is a number k such that $C_n(r_t(e))$ contains k boys $B_1, B_2, \ldots, B_k$ and $k+1$ girls $G_1, G_2, \ldots, G_k, G_{k+1}$ and that B_i knows G_j if and only if they are joined by a straight line segment in Fig. 2. We further

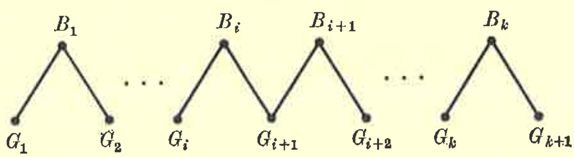

FIG. 2

assume that if $B(r_0(e))$ and $B(r_1(e))$ are in the same community at the beginning of stage n, then $\varphi_e(r_t(e))$ is defined and $B(r_t(e))$ knows $G(\varphi_e(r_t(e)))$ for each $t < 2$ and also that, if $B(r_t(e))$ is B_i and $B(r_{1-t}(e))$ is B_j where $i < j$, then $G(\varphi_e(r_t(e)))$ is G_{i+1} and $G(\varphi_e(r_{1-t}(e)))$ is G_j, so that once the construction is completed, and the community of $B(r_0(e))$ looks like Fig. 1, it will be impossible for φ_e to be a solution to the marriage problem of S.

We assume further that if $e \neq e'$ then $C_n(r_t(e))$ and $C_n(r_{t'}(e'))$ are distinct for any t and t'. Finally, we assume that the community $C_n(r_t(e))$ is always enumerated (as in Fig. 2) in such a way that if $B(r_t(e))$ is B_i then $G_i < G_{i+1}$ (i.e. if G_i is $G(x)$ and G_{i+1} is $G(y)$ then $x < y$) except that in case $B(r_0(e))$ and $B(r_1(e))$ are in the same community then we only make that assumption for $t = 0$.

Stage n. Introduce $B(a)$ to $G(b)$ and $G(b+1)$ and let $r_0(n-1) = a$. Introduce $B(a+1)$ to $G(b+2)$ and $G(b+3)$ and let $r_1(n-1) = a+1$. Then the communities $C_{n+1}(r_t(n-1))$ will be of the correct form.

Suppose that $n = 2^e.q$ where q is odd. If $B(r_0(e))$ and $B(r_1(e))$ are in the same community then we just expand that community by introducing $B(a+2)$ to G_1 and to $G(b+4)$ and $B(a+3)$ to G_{k+1} and to $G(b+5)$. If $B(r_0(e))$ and $B(r_1(e))$ are in different communities, say $C_n(r_0(e))$ has boys $B_1, \ldots, B_k$ and girls $G_1, \ldots, G_{k+1}$ and $C_n(r_1(e))$ has boys $B_1^*, \ldots, B_l^*$ and girls $G_1^*, \ldots, G_{l+1}^*$ then we make the necessary introductions so that the resulting communities are displayed in Fig. 3.

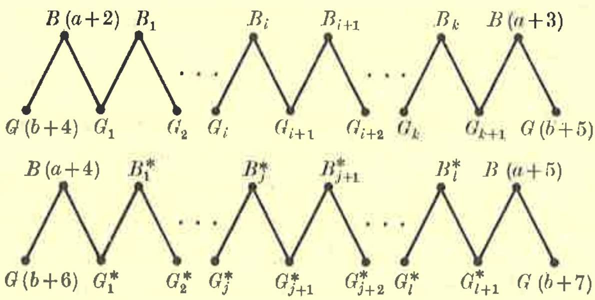

FIG. 3

We now ask whether $\varphi_e^n(r_0(e))$ is defined. $\varphi_e^n(r_1(e))$ is defined, $B(r_0(e))$ knows $G(\varphi_e^n(r_0(e)))$, and $B(r_1(e))$ knows $G(\varphi_e^n(r_1(e)))$. If any of the answers is negative, we can go on to the next stage of the construction. If all of the answers are positive then there is a danger that φ_e may become a solution to the marriage problem of S and that must be prevented.

Suppose then that $B(r_0(e))$ is B_i and that $B(r_1(e))$ is B_j^*. There are four cases.

(a) If G_i is $G(\varphi_e(r_0(e)))$
and if G_j^* is $G(\varphi_e(r_1(e)))$
then introduce $B(a+6)$ to $G(b+4)$ and $G(b+6)$.

(b) If G_i is $G(\varphi_e(r_0(e)))$
and if G_{j+1}^* is $G(\varphi_e(r_1(e)))$
then introduce $B(a+6)$ to $G(b+4)$ and $G(b+7)$.

(c) If G_{i+1} is $G(\varphi_e(r_0(e)))$
and if G_j^* is $G(\varphi_e(r_1(e)))$
then introduce $B(a+6)$ to $G(b+5)$ and $G(b+6)$.

(d) If G_{i+1} is $G(\varphi_e(r_0(e)))$
and if G_{j+1}^* is $G(\varphi_e(r_1(e)))$
then introduce $B(a+6)$ to $G(b+5)$ and $G(b+7)$.

Note that these introductions make the community $C_{n+1}(r_0(e))$ look like Fig. 4 (in case (a)) and that, if this community is expanded only at its ends, no solution can marry both B_i to G_i and B_j^* to G_j^*.

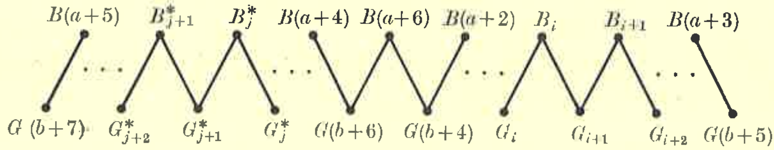

FIG. 4

This completes the nth stage of the construction. Before we proceed to stage $n+1$ we must renumber the people in the communities $C_{n+1}(r_t(e))$ so as to conform with the induction hypothesis. Note that once this is done all of the parts of the induction hypothesis continue to hold at the end of stage n.

This completes the definition of the construction.

We must now show that the society S which results from the construction above is indeed a 2-society, is recursive, has no recursive solutions, but is solvable. That it is a 2-society and is solvable follows from the fact that each community of S has the form described by Fig. 1. That it is recursive follows, as in Theorem 1, from the precautions taken to ensure that two non-strangers are never introduced in the course of the construction. (Again implicit use is made of the enumeration theorem.) That it has no recursive solutions follows from the precautions taken (see Fig. 4) to ensure that not both $B(r_0(e))$ and $B(r_1(e))$ can marry the partners which φ_e would have them marry.

Theorem 3 can be generalized; in fact, it is possible to construct for each k a recursive k-society which is solvable but is not recursively solvable. A proof of this assertion will appear elsewhere.

Now that we have shown that, even if a solvable society is highly recursive, it need not have a recursive solution, it is reasonable to ask whether some other condition, stronger than highly recursive, would guarantee the existence of a recursive solution.

The strongest possible condition of this type can be described in the following way. To say that a society is recursive is to say that any question of the form 'Does $B(n)$ know $G(m)$?' can be answered effectively. To say that a society is highly recursive is to say that, in addition, every question of the form 'Does $B(n)$ know exactly k girls?' can be answered effectively. The strongest possible condition of this type would be that every question about S formulated in the first-order predicate calculus be answered effectively. (A question about S is said to be formulated in the first-order predicate calculus if it can be described using only the formulas $R(t_1, t_2)$ and $t_1 = t_2$, where t_1 and t_2 are either variables or numbers, the logical connectives 'and' ($\bigwedge$), 'or' ($\bigvee$), 'not' ($\sim$), 'if ... then' ($\rightarrow$) and 'if and only if' ($\leftrightarrow$), and the quantifiers 'for all' ($\forall$) and 'there exists' ($\exists$) which range over the natural numbers. Thus the question 'Does $B(n)$ know $G(m)$?' corresponds to $R(2n, 2m+1)$; the question 'Does $B(n)$ know exactly three girls?' corresponds to

$$(\exists v_1)(\exists v_2)(\exists v_3)[v_1 \neq v_2 \wedge v_1 \neq v_3 \wedge v_2 \neq v_3$$

$$\wedge R(2n, v_1) \wedge R(2n, v_2) \wedge R(2n, v_3)$$

$$\wedge (\forall \omega)(R(2n, \omega) \rightarrow (\omega = v_1 \vee \omega = v_2 \vee \omega = v_3))].$$

Similarly questions such as 'Does the community of $B(n)$ have at least 67 members?' and 'Are there three boys who know between them at least six girls?' can be formulated in the first-order predicate calculus.) Such a society is said to be *decidable.*

It is easy to construct societies which are highly recursive but not decidable. For example, if we were to modify the construction above so that when the communities of $B(r_0(e))$ and $B(r_1(e))$ are combined, the remaining ends of the resulting community are appropriately connected, there is no reason to expect that one should be able to determine effectively whether there exists a community with exactly 8,000 people in it.

It is possible to show, using an elimination of quantifiers procedure, that a recursive 2-society in which every community is infinite is in fact decidable; therefore the society constructed in Theorem 3 is decidable. Although the proof of this fact will not be given in detail here, we indicate for those familiar with the elimination of quantifiers procedure what predicates must be adjoined to the language in order to carry out the argument. For each $n > 0$ let '$|x, y| = n$' mean 'there exists a sequence $x = p_0, p_1, \ldots, p_{n-1} = y$ of n different persons, such that for each $i < n-1$, $R(p_i, p_{i+1})$'. Let '$|x, y| > n$' mean '$(x \neq y) \wedge \bigwedge_{0<i\leqslant n} (\sim (|x, y| = i))$'. The addition of the predicates $|x, y| = n$ and $|x, y| > n$ for all n to the language makes it possible to eliminate quantifiers and thereby leads to a decision procedure for the society S.

THEOREM 4. *There is a decidable society S which is solvable but is not recursively solvable.*

Having shown that even a highly recursive society which is solvable need not be recursively solvable it is natural to ask how far from recursive its solutions may be.

THEOREM 5. *Let S be a highly recursive society which is solvable. Then S has a solution whose degree of unsolvability does not exceed* $\mathbf{0}'$.

Proof. For each n let S_n be the partial society

$$\{B(i) \mid i \leqslant n\} \cup \bigcup_{i \leqslant n} \{G(j) \mid B(i) \text{ knows } G(j)\}.$$

By assumption each S_n is solvable, so if we let T_n be the set of all solutions to the marriage problem of S_n then $T_n \neq \emptyset$. Let $T = \bigcup_{n \in N} T_n$ and define a partial ordering on T by $f \leqslant g$ if g extends f. With this definition T is a tree which branches finitely and which has an infinite number of nodes.

Thus König's lemma applies. (See [**12**], p. 165, or another book on Set Theory for a discussion of this result.) Our proof consists of a careful analysis of the proof of König's lemma, since finding a solution to the marriage problem of S is equivalent to constructing an infinite path in T.

The proof of König's lemma starts by finding an element f_0 in T_0 which has infinitely many elements of T extending it; this is possible since there are only finitely many elements of T_0 and infinitely many elements in T each of which extends some element of T_0. Proceeding inductively we assume that we have defined an element $f_n \in T_n$ which has infinitely many elements of T extending it; we now look at the finitely many elements of T_{n+1} extending it and, as above, find an element f_{n+1} in T_{n+1} which extends f_n and which has infinitely many extensions in T.

To apply recursion-theoretic techniques to this proof we must first specify which element of T_{n+1} we choose to be f_{n+1} if there is more than one which extends f_n and has infinitely many extensions in T. Let us specify that f_{n+1} is that element of T_{n+1} which has these properties and which has the smallest possible value at $n+1$.

Let f be the (unique) extension of all the f_n. Clearly f is a solution to the marriage problem of S. We shall show that this function f has the desired property.

For consider the following property of the numbers z, s, and n: 'that s codes a sequence of $n+1$ numbers $s_0, s_1, s_2, \ldots, s_n$; that a solution to the marriage problem of S_n would consist of marrying $B(0)$ to $G(s_0)$, $B(1)$ to $G(s_1), \ldots, B(n)$ to $G(s_n)$; that marrying, in addition, $B(n+1)$ to $G(z)$ would be a solution to the marriage problem of S_{n+1}, and that this solution extends to a solution of S_m for every $m > n$'. To say that this solution extends to a solution of S_m for every $m > n$ is to say that 'for each such m there is a sequence of $m+1$ numbers $s_0, s_1, \ldots, s_n, z, t_{n+2}, t_{n+3}, \ldots, t_m$ such that a solution to the marriage problem of S_m would consist of marrying $B(0)$ to $G(s_0)$, ..., $B(a)$ to $G(s_n)$, $B(n+1)$ to $G(z)$, $B(n+2)$ to $G(t_{n+2})$, ..., $B(m)$ to $G(t_m)$'. If we write this predicate out using the formalism of number theory we obtain the following expression:

$$\begin{aligned}
&(\forall\, i)_{i<\mathrm{lh}\, s} R(B(i), G((s)_i) \wedge (\forall\, i)_{i<\mathrm{lh}\, s} (\forall\, j)_{j<i}((s)_i \neq (s)_j) \\
&\wedge (\forall\, i)_{i<\mathrm{lh}\, s}((s)_i \neq z) \wedge R(B(n+1), G(z)) \\
&\wedge \mathrm{lh}\; s = n+1 \\
&\wedge (\forall\, m)\,[m > \mathrm{lh}\; s \rightarrow (\exists\, t)\,(\mathrm{lh}\; t = m \wedge \\
&\quad (\forall\, i)_{i<\mathrm{lh}\, s}((s)_i = (t)_i) \wedge (t)_{\mathrm{lh}\, s} = z \wedge \\
&\quad (\forall\, i)_{i<m}\,(R(B(i), G((t)_i)) \wedge \\
&\quad (\forall\, i)_{i<m}\,(\forall\, j)_{j<i}\,((t)_i \neq (t)_j))].
\end{aligned}$$

Using standard procedures for moving quantifiers to the front (see [**1**], p. 185, [**9**], pp. 85–90, or [**14**], § 14.3), the above can be written as $(\forall\, m)\,(\exists\, t)\, R(m, t, s, z, n)$ which has one quantifier too many for our purposes. However, we have not yet used the assumption that S is highly recursive. This assumption enables us to give a recursive bound for the existential quantifier; that is, the fact that the society is highly recursive enables us to decide how long to search for a solution to the marriage problem for S_m which extends the given one for S_{n+1} before we give up, and it tells us that if we give up the search without finding such a solution, we do so because there is none.

Specifically, given $m > \mathrm{lh}\; s$ we need find, for each $i < m$, only the value of $h(i)$, then find, for each $i < m$, the $h(i)$ girls whom $B(i)$ knows, for then we shall have enough information on hand to determine whether there is a solution to the marriage problem of S_m extending the given solution of S_{n+1}.

Define $g(i) = \mu t\,(\sum_{j<t} X_R(B(i), G(j)) = h(i))$. Since h is recursive, so is g and every girl that $B(i)$ knows is among $G(0), G(1), \ldots, G(g(i))$. Hence $(t)_i$ will be at most $g(i)$ for each i. Hence we may assume that t is at most $\tilde{g}(m)$. Thus the predicate above can be written in the form

$$(\forall\, m)\,(\exists\, t)_{t<\tilde{g}(m)}\, R(m, t, s, z, n)$$

which in turn is of the form $(\forall\, m) Q(m, s, z, n)$ where Q is a recursive predicate.

We now define the function f by a course-of-values recursion:

$$f(n) = \mu z\,[(\forall\, m)\, Q(m, \bar{f}(n), z, n \dot{-} 1)].$$

It is clear that this function f is the solution referred to earlier. It is also clear that f is recursive in a predicate whose degree is less than or equal to $\mathbf{0}'$. Hence the degree of f is $\leqslant \mathbf{0}'$.

We can strengthen Theorem 5 considerably, but more powerful recursion-theoretic techniques are necessary; we have left both statement and proof for later (Theorem 11).

We now return to a consideration of the other condition imposed on a recursive society S in Theorem 2—namely that S have a unique solution to its marriage problem. As we observed earlier, the existence of a unique solution to the marriage problem of the recursive society S is not sufficient to guarantee that S be recursively solvable. We wish to discuss at this point what can be said about such a society. The next two theorems show that there is a recursive society with a unique solution and that solution has degree of unsolvability $\mathbf{d}$ if and only if $\mathbf{d} \leqslant \mathbf{0}'$.

THEOREM 6. *Let S be a recursive society which has a unique solution to its marriage problem. Let $\mathbf{d}$ be the degree of unsolvability of that solution. Then $\mathbf{d} \leqslant \mathbf{0}'$.*

Proof. We remind the reader of the proof of Theorem 2, where, with the additional hypothesis that S be highly recursive, we proved that the unique solution must be recursive. It is easy to see that what was actually proved was that the unique solution is recursive in the function h which there was recursive. Thus we need to show only that in general the function h is recursive in $\mathbf{0}'$.

Define $g(i) = (\mu t)((\forall j)(j > t \rightarrow {\sim} R(B(i), G(j))))$. Then g is recursive in $\mathbf{0}'$. Also

$$h(i) = \sum_{j<g(i)} X_R(B(i), G(j))$$

so that h is recursive in g. Hence h, and therefore the unique solution, is recursive in $\mathbf{0}'$.

COROLLARY. *Let S be a recursive society which has a finite number of solutions to its marriage problem. Then each of its solutions is recursive in $\mathbf{0}'$.*

Proof. We proceed as in the proof of the corollary to Theorem 2 and instead of applying Theorem 2 apply Theorem 6.

THEOREM 7. *If $\mathbf{d} \leqslant \mathbf{0}'$, then there is a recursive society which has a unique solution and that solution has degree of unsolvability $\mathbf{d}$.*

Proof. Let D be a set whose degree of unsolvability is $\mathbf{d}$; then since $\mathbf{d} \leqslant \mathbf{0}'$ there is a number c such that $X_D = \varphi_c^K$.

We define simultaneously the society S and three one-to-one recursive functions r, s_0, and s_1, where the ranges of s_0 and s_1 are disjoint; at the end of stage n, $r(i)$, $s_0(i)$, and $s_1(i)$ will be defined for all $i < n$.

Intuitively the construction will guarantee that $B(r(e))$ will marry $G(s_0(e))$ if and only if $e \in D$ and that $B(r(e))$ will marry $G(s_1(e))$ if and only if $e \notin D$. This will imply that the unique solution f has degree **d**.

We assume as part of the inductive hypotheses that for each $e < n$ there is a natural number k such that the community $C_n(r(e))$ contains k boys $B_1, B_2, \ldots, B_k$ and k girls $G_1, G_2, \ldots, G_k$ and that, for some $t < 2$, B_i knows G_j if and only if they are joined by a straight line segment in Fig. $1'$ below.

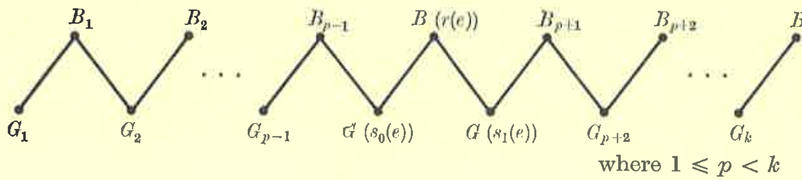

FIG. $0'$

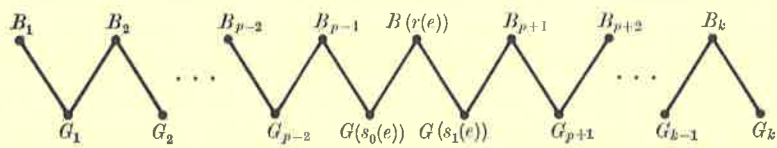

FIG. $1'$

Note that if $C_n(r(e))$ is stable then, in any solution to the marriage problem of S, $B(r(e))$ marries $G(s_0(e))$ if and only if $C_n(r(e))$ looks like Fig. $0'$, and $B(r(e))$ marries $G(s_1(e))$ if and only if $C_n(r(e))$ looks like Fig. $1'$.

Define a recursive function

$$X(t,e) = \begin{cases} 1 & \text{if } \varphi_c^{K,t}(e) \simeq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $X(t,e)$ is (approximately) the value of $\varphi_c^K(e)$ after t steps—or, since φ_c^K is the characteristic function of the set D of degree **d**, $X(t,e) = 1$ if and only if the number e appears to be in the set D after t steps. As t increases, for each fixed e, the number e may appear alternately to be in D and to be in the complement of D, but eventually we will reach a point (although we have no way of knowing whether we have reached that point) where the appearance becomes the reality. We have arranged matters below so that if e eventually ends up in D, then $C_n(r(e))$ will become stable in the form of Fig. 0′, so that $B(r(e))$ must marry $G(s_0(e))$—and if e eventually ends up in the complement of D, then $C_n(r(e))$ will become stable in the form of Fig. 1′, so that $B(r(e))$ must marry $G(s_1(e))$.

Stage n. Let $r(n-1) = a$, $s_0(n-1) = b$, $s_1(n-1) = b+1$; introduce $B(a)$ to $G(b)$ and $G(b+1)$ and introduce $B(a+1)$ to $G(b+1)$.

Suppose that $n = 2^e.q$, where q is odd. Then if $X(n,e) = 1$ we want $C_{n+1}(r(e))$ to look like Fig. 0′ whereas if $X(n,e) = 0$ we want $C_{n+1}(r(e))$ to look like Fig. 1′. Hence if $X(n, e) = 1$ and $C_n(r(e))$ looks like Fig. 0′ *or* if $X(n,e) = 0$ and $C_n(r(e))$ looks like Fig. 1′ we proceed directly to the next stage. If, however, $X(n,e) = 1$ and $C_n(r(e))$ looks like Fig. 1′, then we introduce B_1 to $G(b+2)$ and G_k to $B(a+2)$. Similarly, if $X(n,e) = 0$ and $C_n(r(e))$ looks like Fig. 0′, then we introduce G_1 to $B(a+2)$ and B_k to $G(b+2)$.

This completes the description of the construction. We must now show that

(i) S is recursive,

(ii) S has a unique solution f,

(iii) f has degree **d**.

As to (i), we proceed as in Theorem 1.

As to (ii), it suffices to show that for each n, $C_n(r(e))$ is stable for sufficiently large n, since each $C_n(r(e))$ has a unique solution to its marriage problem. But $X(t,e) = X_D(e)$ for t beyond some number $t_0(e)$, so that $C_t(r(e)) = C_{t_0(e)}(r(e))$ for all $t \geqslant t_0(e)$.

As to (iii), it is clear that D is recursive in f since $X_D(e) = 1$ if and only if f marries $B(r(e))$ to $G(s_0(e))$. On the other hand, given $B(j)$, we can effectively determine which $B(r(e))$ is in his community and which two girls he knows. Then using our knowledge of whether $e \in D$ or not we can determine whom $B(r(e))$ will marry. Using that information, we can determine effectively whom $B(j)$ will marry. Hence f is recursive in D. Hence f has degree **d**.

Theorems 5 and 6 state that if a solvable recursive society either is highly recursive or has a unique solution then that solution is recursive in $\mathbf{0}'$. A reasonable conjecture would be that any solvable recursive society has a solution recursive in $\mathbf{0}'$. This conjecture is false.

THEOREM 8. *There exists a recursive society which is solvable but which has no solution recursive in* $\mathbf{0}'$.

Proof. We define simultaneously the society S and three one-to-one recursive functions r, s_0, and s_1; at the end of stage n, $r(i)$, $s_0(i)$, and $s_1(i)$ will be defined for all $i < n$.

Intuitively the construction will guarantee that if $\varphi_e^K(r(e))$ is defined then $B(r(e))$ cannot marry $G(\varphi_e^K(r(e)))$ and thus that φ_e^K cannot be a solution to the marriage problem of S.

We include as part of the inductive hypothesis the same assumptions made about the communities $C_n(e)$ as were made in the proof of Theorem 7.

Stage n. Let $r(n-1) = a$, $s_0(n-1) = b$, $s_1(n-1) = b+1$; introduce $B(a)$ to $G(b)$ and $G(b+1)$ and introduce $B(a+1)$ to $G(b+1)$.

Suppose that $n = 2^e.q$, where q is odd. Then if $\varphi_e^{K,n}(r(e))$ is defined and equals $s_t(e)$ we want $C_{n+1}(r(e))$ to look like Fig. $(1-t)'$ of Theorem 7. Hence, if $\varphi_e^{K,n}(r(e)) = s_0(e)$ and $C_n(r(e))$ looks like Fig. $0'$ then we introduce G_1 to $B(a+2)$ and B_k to $G(b+2)$. Similarly if $\varphi_e^{K,n}(r(e)) = s_1(e)$ and $C_n(r(e))$ looks like Fig. $1'$ we introduce B_1 to $G(b+2)$ and G_k to $B(a+2)$. Otherwise we proceed directly to the next stage.

This completes the description of the construction. We must now show that

(i) S is recursive,

(ii) S is solvable,

(iii) S has no solution recursive in $\mathbf{0}'$.

As to (i), we proceed as in Theorem 1.

As to (ii), we note that each community of S either looks like Fig. $0'$ or $1'$ or, as a result of the eternal oscillation of $\varphi_e^{K,n}(r(e))$, looks like Fig. 5. Thus

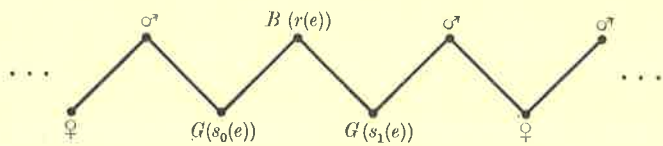

FIG. 5

each community of S has a solution to its marriage problem, and therefore so does S.

As to (iii), if f is recursive in $\mathbf{0}'$ then $f = \varphi_e^K$ for some e. Hence for n beyond some number n_0, $f(r(e)) = \varphi_e^{K,n}(r(e))$. Hence $C_{n_0+1}(e)$ is stable and $B(r(e))$ cannot marry $G(\varphi_e^K(r(e)))$. Therefore f is not a solution to the marriage problem of S.

In spite of the result of Theorem 8, it is possible to find a degree such that every solvable recursive society has a solution recursive in that degree. In fact, this was essentially done in the proof of Theorem 5.

THEOREM 9. *Let S be a solvable recursive society. Then S has a solution which is recursive in* $\mathbf{0}''$.

Proof. The proof of Theorem 5 shows that if S is any solvable recursive society then S has a solution which is recursive in the predicate $(\forall m)(\exists t)R(m,t,s,z,n)$, where R is recursive. (The hypothesis that S be highly recursive was then used to eliminate one of the quantifiers.) Hence there is a solution to the marriage problem of S which is recursive in $\mathbf{0}''$.

Theorems 8 and 9 leave between them a wide gap. This gap is partially closed by the following theorem which is patterned on the basis theorem of Soare and Jockusch ([7]). In a similar way the theorem after that partially closes the gap between Theorems 3 and 5.

THEOREM 10. *If the recursive society S is solvable then it has a solution whose degree* $\mathbf{d}$ *satisfies* $\mathbf{d}' \leqslant \mathbf{0}''$.

Proof. This improvement of Theorem 9 is essentially the device used by Soare and Jockusch ([7]) to improve the basis theorems of Kreisel and Shoenfield ([**15**]).

In the proof we try to find a solution f recursive in $\mathbf{0}''$ so that the questions 'Is $\varphi_e^f(e)$ defined?' of degree $\mathbf{d}'$ can be answered recursively in $\mathbf{0}''$.

We shall proceed as in the proof of Theorem 5 to find a sequence of partial functions $\{f_n \mid n \in N\}$ such that each f_n is a solution to the marriage problem of S_n. Simultaneously we shall define a strictly increasing function r such that $\varphi_e^f(e)$ is undefined if and only if e is in the range of r, where $f = \bigcup_{n \in N} f_n$. Both f and r will be recursive in $\mathbf{0}''$ and so, since the fact that r is a strictly increasing function recursive in $\mathbf{0}''$ implies that its range is also recursive in $\mathbf{0}''$ the questions 'Is $\varphi_e^f(e)$ defined?' can be answered recursively in $\mathbf{0}''$.

We shall use the terminology developed in the proof of Theorem 5. However, we shall convert a function $f_n \in T_n$ into a number t such that $(t)_i = f_n(i)$ for each $i \leqslant n$. We say that a $t \in T$ is *acceptable for* e if $\varphi_e^t(e)$ is undefined. Thus if t is considered as an approximation to the solution f then t is acceptable for e approximately if $\varphi_e^f(e)$ is to be undefined.

We assume as part of the inductive hypothesis that at stage n we have defined $f_n \in T_n$ and $r(i)$ for all $i \leqslant n$. We assume further that f_n has infinitely many extensions in T which are acceptable for every $r(i)$, where $i \leqslant n$.

Now define $r(n+1)$ to be the least $e > r(n)$ such that there are infinitely many extensions of f_n which are acceptable for e as well as for all $r(i)$ for $i \leqslant n$. (Such an e always exists since there are infinitely many e such that $\varphi_e^g(e)$ is undefined for every g.) Let f_{n+1} be an extension of f_n, $f_{n+1} \in T_{n+1}$, such that f_{n+1} has infinitely many extensions which are acceptable for each $r(i)$ for $i \leqslant n+1$, and such that $f_{n+1}(n+1)$ is minimal. Here again we are using König's lemma.

It is clear, simply from the fact that $f_n \in T_n$ and f_{n+1} extends f_n for every n, that $\bigcup_{n \in N} f_n$ is indeed a solution to the marriage problem of S. We must show that both f and r are recursive in $\mathbf{0}''$ in order to conclude that the degree $\mathbf{d}$ of f satisfies $\mathbf{d}' \leqslant \mathbf{0}''$.

Define the predicate $N(m,t,s,e)$ to mean that t is a proper extension of s which codes a solution to the marriage problem of S_m and such that $\varphi_e^t(e)$ is undefined. This predicate is recursive since it can be written formally as

$$\begin{aligned}&\mathrm{lh}\ s < m \wedge \mathrm{lh}\ t = m \wedge (\forall\, i)_{i<\mathrm{lh}\, s}((t)_i = (s)_i) \wedge \\ &(\forall\, i)_{i<m}\,[R(B(i), G((t)_i)) \wedge (\forall\, j)_{j<i}\,((t)_i \neq (t)_j)] \wedge \\ &\varphi_e^t(e) \text{ is undefined.}\end{aligned}$$

(Note that to say that $\varphi_e^t(e)$ is defined is to say that it is defined in at most lh t steps, so that the answer to the question 'Is $\varphi_e^t(e)$ defined?' can be recursively obtained.)

Using N we define the predicate $M(m,t,s,e,q)$ to mean that in addition t is acceptable for each $(q)_i$, where $i <$ lh q. Thus $M(m,t,s,e,q)$ is recursive since it can be defined formally by

$$N(m,t,s,e) \wedge (\forall\, i)_{i<\mathrm{lh}\, q} N(m,t,s,(q)_i).$$

Define the predicate $P(s,e,q)$ to mean that for every $m >$ lh s there is an extension of s which is a solution to the marriage problem of S_m and which is acceptable to e and to all $(q)_i$ for $i <$ lh q. Formally, $P(s,e,q)$ is the predicate

$$(\forall\, m)\,(m > \mathrm{lh}\ s \rightarrow (\exists\, t)\, M(m,t,s,e,q)).$$

Since the predicate P can be written in the form $(\forall\, m)\,(\exists\, t)\, F(m,t,s,e,q)$, where F is recursive, P is itself recursive in $\mathbf{0}''$.

We now define simultaneously by induction two functions as follows:

$$\begin{aligned}r(n) &= \mu e\,[P(\tilde{f}(n), e, \tilde{r}(n)) \wedge e > (\tilde{r}(n))_{n\dot{-}1}],\\ f(n) &= \mu z\,[P(\tilde{f}(n).p_n{}^z, r(n), \tilde{r}(n))].\end{aligned}$$

Thus $r(n)$ is the smallest e such that for every $m > n$ there is an extension t of f_{n-1} which is a solution to the marriage problem of B_{m-1} and which is acceptable for e and for all $r(i)$ for $i < n$. As noted above, we know that such an e must exist. Also $f(n)$ is defined here as the least z such that by extending the solution f_{n-1} to a solution for S_n by marrying $B(n)$ to $G(f(n))$ we still have infinitely many extensions which are acceptable for all $r(i)$ where $i \leqslant n$.

Hence the f and r defined here are formally and precisely the same as the f and r defined in the construction above. But f and r here are both recursive in $\mathbf{0}''$.

To see that the questions 'Is $\varphi_e^f(e)$ defined?' can indeed be answered recursively in $\mathbf{0}''$, we note that, first of all, if e is in the range of r then clearly $\varphi_e^{f_n}(e)$ is undefined for each n so that $\varphi_e^f(e)$ is undefined. Conversely if e_0 is not in the range of r, say $r(n-1) < e_0 < r(n)$ (recall that r is strictly increasing) then $\sim P(\tilde{f}(n), e_0, \tilde{r}(n))$ so that

$$(\exists\, m)\,(m > n \wedge (\forall\, t)\,[(\forall\, i)_{i<n} N(m,t,\tilde{f}(n),(r)_i) \to \sim N(m,t,\tilde{f}(n),e_0)]).$$

This says that once f_m has been defined, as long as this is done in accordance with our plan, then at that point $\varphi_e^{f_m}(e)$ will be defined, so that $\varphi_e^f(e)$ will be defined.

Hence $\varphi_e^f(e)$ is defined if and only if e is not in the range of r if and only if $(\forall\, i)_{i<e}(r(i) \neq e)$. Thus the question of whether $\varphi_e^f(e)$ is defined is recursive in r and hence in $\mathbf{0}''$. Thus the degree $\mathbf{d}$ of f satisfies $\mathbf{d}' \leqslant \mathbf{0}''$.

THEOREM 11. *Let S be a highly recursive society which is solvable. Then S has a solution whose degree $\mathbf{d}$ satisfies $\mathbf{d}' = \mathbf{0}'$.*

Proof. We use the proof above, observing as at the end of the proof of Theorem 5, that the fact that S is highly recursive enables us to conclude that the predicate $P(s,e,q)$ defined above as $(\forall\, m)\,(\exists\, t)\,F(m,t,s,e,q)$ can be written as $(\forall\, m)\,(\exists\, t)_{t<\tilde{g}(m)}\,F(m,t,s,e,q)$, where g is recursive, so that P is recursive in $\mathbf{0}'$. Since the functions f and r are recursive in P, $\mathbf{d}' \leqslant \mathbf{0}'$. Of course this implies that $\mathbf{d}' = \mathbf{0}'$.

3. We remarked at the beginning of the preceding section that it is possible to have a solution to the marriage problem for a certain society in which some of the girls remain unmarried; this is possible because we demanded only that in a solution every boy be married. In this section we consider the symmetric version of the marriage problem.

If R is a partial society and f is a function whose domain is $\{n \mid B(n)$ is in the field of $R\}$, then f is called a *solution to the symmetric marriage problem of R* if f is one-to-one, if f is onto $\{m \mid G(m)$ is in the field of $R\}$, and if $B(n)$ knows $G(f(n))$ for every n in the domain of f. The partial society R is said to be *symmetrically solvable* if there is a solution to the symmetric marriage problem of R.

A further generalization of the theorem of Philip Hall is the statement that a partial society is symmetrically solvable if and only if every n boys know among them at least n girls and every n girls know among them at least n boys. (See [**11**], p. 536.)

It is evident that a partial society is symmetrically solvable if and only if each of its communities is symmetrically solvable; it has a unique symmetric solution if and only if each of its communities has a unique symmetric solution.

A society S is *recursively symmetrically solvable* if there is a recursive function f which is a solution to the symmetric marriage problem of S.

We proceed to give the symmetric versions of Theorems 1–11 above.

THEOREM 1*. *There is a recursive society S which is symmetrically solvable but is not recursively symmetrically solvable.*

Proof. The society constructed in the proof of Theorem 1 is symmetrically solvable.

A recursive society S is said to be *highly* recursive* if there is a recursive function h^* such that, for each i, the number of people whom i knows is exactly $h^*(i)$.

THEOREM 2*. *If S is a highly* recursive society and the symmetric marriage problem for S has a unique solution, then that solution is recursive.*

Proof. The proof is a symmetrized version of the proof of Theorem 2.

We shall describe an algorithm which the matchmaker can use to find the symmetric solution to the symmetric marriage problem of S. The fact that the algorithm works is a consequence of the fact that

(∗) if R is a partial society which has a unique solution to its symmetric marriage problem then some person in the society must know exactly one person of the opposite sex.

This statement is a consequence of the lemma which is proved below.

The matchmaker proceeds as in the proof of Theorem 2 to compute $h^*(0), h^*(1), \ldots$ until he finds a person who knows exactly one person of the opposite sex, finds that person's mate, deletes the couple from the society, and starts the procedure over with the partial society thus obtained. It is then clear that if he repeats this procedure for ever, at each stage he pairs off a couple who in the unique symmetric solution must be married, at each stage he is able to continue because of (∗), and, using (∗) just as in the proof of Theorem 2, every person is paired off at some time in the course of the algorithm.

We mention, for emphasis, that at each point in the proof of Theorem 2 where boys are treated asymmetrically, a symmetric version of that fragment of the argument can and should be substituted and the result will be a proof of Theorem 2*.

COROLLARY*: *Let S be a highly* recursive society whose symmetric marriage problem has a finite number of solutions. Then these solutions are all recursive.*

The lemma on which the Theorem 2* above rests is

LEMMA*. *Let R be a partial society in which each person knows at least two people of the opposite sex. If R is symmetrically solvable, then the symmetric marriage problem of R has at least two solutions.*

Proof. Let f be a solution to the symmetric marriage problem of R. Let $B(i_0)$ be any boy in R and let $f(i_0) = j_0$ so that the solution f marries $B(i_0)$ to $G(j_0)$. Let $G(j_1)$ be another girl that $B(i_0)$ knows and let $B(i_{-1})$ be another boy that $G(j_0)$ knows. Let $B(i_1)$ be the boy whom f marries to $G(j_1)$. If $B(i_1)$ is $B(i_{-1})$ then we can define a new symmetric solution f^* which marries $B(i_1)$ to $G(j_0)$ and $B(i_0)$ to $G(j_1)$ and each other boy $B(i)$ to $G(f(i))$. So we may assume that $B(i_1)$ is different from $B(i_{-1})$. Let $G(j_{-1})$ be the girl whom f marries to $B(i_{-1})$; then $G(j_{-1})$ is different from both $G(j_0)$ and $G(j_1)$. We are now faced with the situation of Fig. 6.

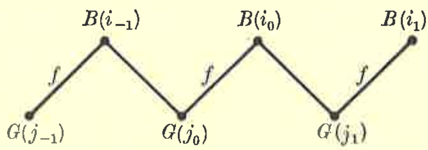

FIG. 6

where a line between two people indicates that they know each other and an f on a line indicates that the couple is married in the solution f.

Assume now that we have continued the definition and found $2n+1$ distinct boys and $2n+1$ distinct girls so that we have obtained Fig. 7.

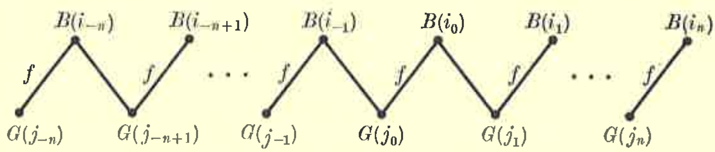

FIG. 7

Let $G(j_{n+1})$ be another girl whom $B(i_n)$ knows. If $G(j_{n+1})$ is $G(j_t)$ for some t, $-n \leqslant t < n$, then we can define a new symmetric solution f^* which marries $B(i_n)$ to $G(j_t)$, $B(i_r)$ to $G(i_{r+1})$ for $t \leqslant r < n$, and each other boy $B(i)$ to $G(f(i))$. So we may assume that $G(j_{n+1})$ is distinct from the $2n+1$ girls above. Let $B(i_{n+1})$ be the boy to whom f marries $G(j_{n+1})$. Let $B(i_{-n-1})$ be another boy whom $G(j_{-n})$ knows. If he is $B(i_t)$ for some t, $-n < t \leqslant n+1$, then again we can define a symmetric solution f^*. So we may assume that $B(i_{-n-1})$ is distinct from the $2n+2$ boys already determined. Let $G(j_{-n-1})$ be the girl to whom f marries $B(i_{-n-1})$. We have thus obtained the diagram above, with $n+1$ replacing n.

Thus, proceeding inductively, either we find a new symmetric solution or we find a set $\{B(i_t) \mid t \in Z\}$ of boys and a set $\{G(j_t) \mid t \in Z\}$ (here Z is the set of integers) such that, for each t, $B(i_t)$ knows $G(j_t)$ and $G(j_{t+1})$ and is married to $G(j_t)$ in the solution f. We can now define a new symmetric solution f^* by stipulating that, for each t, f^* marries $B(i_t)$ to $G(j_{t+1})$ and each other boy $B(i)$ to $G(f(i))$. We have thus obtained a second symmetric solution to the marriage problem of S.

It is reasonable to ask whether Theorem 2* would be true assuming that S was highly recursive (but not highly* recursive). It is perhaps too much to expect that the conclusion can be symmetrized without that happening to the hypotheses as well. This is borne out by the following theorem (which is not numbered so as not to destroy the parallel numberings of the previous section and this one).

THEOREM *. *There is a highly recursive society which has a unique symmetric solution but is not recursively symmetrically solvable. In fact for each degree* $\mathbf{d}$, *such that* $\mathbf{0} < \mathbf{d} \leqslant \mathbf{0}'$ *ther is a highly recursive society which has a unique symmetric solution and that solution has degree* $\mathbf{d}$.

Proof. The construction is the same as that in the proof of Theorem 7, except that a small modification is needed. As a result of this modification, the society no longer has a unique solution to its marriage problem; however, it still has a unique symmetric solution. This solution has degree $\mathbf{d}$ and, furthermore, every boy in the society knows exactly two girls, so that the society is trivially highly recursive.

The modification can be described as follows. If at a certain stage n we are dealing with the community $C_n(r(e))$, if that community looks like Fig. 0′, and if there is no reason to alter the community so that it looks like Fig. 1′, then at stage n we introduce B_k to a new girl and introduce her to a new boy; similarly, if the community looks like Fig. 1, and if there is no reason to alter it so that it looks like Fig. 0, then at stage n we introduce B_1 to a new girl and introduce her to a new boy.

It is easy to verify that the society which results has all the required properties.

We proceed with our analysis of the symmetric versions of Theorems 1–11.

We ask, analogously to the preceding section, whether the assumption of highly* recursive is sufficient to guarantee that a symmetrically solvable society be recursively symmetrically solvable. Again the answer is ‘No’, and in fact the counter-example constructed in Theorem 3, being a 2-society, is highly* recursive. Thus the statements below need no further comment.

THEOREM 3*. *There exists a recursive 2-society S which is symmetrically solvable but not recursively symmetrically solvable.*

THEOREM 4*. *There is a decidable society S which is symmetrically solvable but is not recursively symmetrically solvable.*

Having shown that even a highly* recursive society which is symmetrically solvable need not be recursively solvable, it is natural to ask how far from recursive its symmetric solutions need be.

THEOREM 5*. *Let S be a highly* recursive society which is symmetrically solvable. Then S has a symmetric solution whose degree does not exceed $\mathbf{0}'$.*

Proof. The proof will be a symmetric version of the proof of Theorem 5. For each n let S_n^* be the partial society

$$\{B(i)\,|\,i < n\} \cup \{G(i)\,|\,i < n\} \cup \bigcup_{i \leqslant n} \{G(j)\,|\,B(i) \text{ knows } G(j)\}$$
$$\cup \bigcup_{i \leqslant n} \{B(j)\,|\,B(j) \text{ knows } G(i)\}.$$

By assumption each S_n^* has a solution to its marriage problem in which each $B(i)$ and $G(i)$ are married, for each $i \leqslant n$. From each such solution we can extract an ordered pair of one-to-one functions $\langle f, g\rangle$ called a *semi-solution* for S_n^* whose domains are both $\{i\,|\,i \leqslant n\}$ with the additional properties that for each $i \leqslant n$, $B(i)$ knows $G(f(i))$ and $G(i)$ knows $B(g(i))$, and that f and g are compatible in the sense that if $f(i) \leqslant n$ for some i then $g(f(i)) = i$ and if $g(i) \leqslant n$ for some i then $f(g(i)) = i$. Thus if we let T_n^* be the set of all such ordered pairs of functions, that is, all semi-solutions for S_n^*, the hypotheses of the theorem imply that $T_n^* \neq \emptyset$. Let $T^* = \bigcup_{n \in N} T_n^*$ and define a partial ordering on T^* by $\langle f, g\rangle \leqslant \langle f', g'\rangle$ if f' extends f and g' extends g. With this definition T^* is a tree which branches finitely and has an infinite number of nodes.

As in the proof of Theorem 5, we now apply recursion-theoretic techniques to the proof of König's lemma. That is, we define a sequence $(\langle f_n, g_n\rangle \in T_n^*\,|\,n \in N)$ so that if $m \leqslant n$ then $\langle f_m, g_m\rangle \leqslant \langle f_n, g_n\rangle$. Assuming that $\langle f_n, g_n\rangle$ has been defined and satisfies the suitable inductive hypotheses, we define $\langle f_{n+1}, g_{n+1}\rangle$ to be the element of T_{n+1}^* which extends $\langle f_n, g_n\rangle$ and which yields the smallest value for $f(n+1)$ and, in case of a tie, the smallest value for $g(n+1)$ also.

Let f be the unique extension of all the f_n and let g be the unique extension of all the g_n. It is clear from the construction that a symmetric solution to the marriage problem of the society consists in marrying each boy $B(i)$ to $G(f(i))$, or equivalently each girl $G(i)$ to $B(g(i))$.

We show now that this solution is in fact recursive in $\mathbf{0}'$.

Consider the following property of the numbers z, w, s, t, and n: 'that s codes a sequence of $n+1$ distinct numbers $s_0, s_1, \ldots, s_n$; that t codes a sequence of $n+1$ distinct numbers $t_0, t_1, \ldots, t_n$; that for each $i \leqslant n$, $B(i)$ knows $G(s_i)$ and $G(i)$ knows $B(t_i)$; if $s_i \leqslant n$ then $t_{s_i} = i$ and if $t_i \leqslant n$ then $s_{t_i} = i$ (so that $\langle s, t\rangle$ can be thought of as a semi-solution for S_n^*); that marrying, in addition, $B(z)$ to $G(w)$ would be a semi-solution for S_{n+1}^*, and that this semi-solution extends to a semi-solution for S_m^* for every $m > n$'.

If we write this predicate out using the formalism of number theory we obtain the following expression:

$$
\begin{aligned}
&\mathrm{lh}\, s = n+1 \wedge (\forall i)_{i<\mathrm{lh}\, s} (R(B(i), G((s)_i))) \wedge \\
&\mathrm{lh}\, t = n+1 \wedge (\forall i)_{i<\mathrm{lh}\, t} R(B((t)_i), G(i)) \wedge \\
&(\forall i)_{i<\mathrm{lh}\, s} (\forall j)_{j<i} ((s)_i \neq (s)_j \wedge (t)_i \neq (t)_j) \wedge \\
&(\forall i)_{i<\mathrm{lh}\, s} ((s)_i \leqslant n \to (t)_{(s)_i} = i. \wedge .(t)_i \leqslant n \to (s)_{(t)_i} = i) \wedge \\
&(\forall i)_{i<\mathrm{lh}\, s} ((s)_i \neq z \wedge (t)_i \neq w) \wedge \\
&(\forall i)_{i<\mathrm{lh}\, s} ((s)_i = n+1 \to w = i. \wedge .(t)_i = n+1 \to z = i) \wedge \\
&R(B(n+1), G(z)) \wedge R(G(n+1), B(w)) \wedge \\
&(\forall m) (m > \mathrm{lh}\, s \to (\exists s') (\exists t') (\mathrm{lh}\, s' = m \wedge \\
&\qquad \mathrm{lh}\, t' = m \wedge (\forall i)_{i<\mathrm{lh}\, s} ((s)_i = (s')_i \wedge (t)_i = (t')_i) \wedge \\
&\qquad (s')_{n+1} = z \wedge (t')_{n+1} = w \wedge \\
&\qquad (\forall i)_{i<m} (R(B(i), G((s')_i)) \wedge R(B((t')_i), G(i))) \wedge \\
&\qquad (\forall i)_{i<m} (\forall j)_{j<i} ((s')_i \neq (s')_j \wedge (t')_i \neq (t')_j) \wedge \\
&\qquad (\forall i)_{i<m} ((s')_i < m \to (t')_{(s')_i} = i. \wedge .(t')_i < m \to (s')_{(t')_i} = i)))
\end{aligned}
$$

As in the proof of Theorem 5, this expression can be written as $(\forall m) (\exists s') (\exists t') R^*(m, s', t', s, t, z, w, n)$, where R^* is recursive. As before we now apply the assumption that the society is highly* recursive to get recursive bounds for the existential quantifiers. Specifically, if $m > \mathrm{lh}\, s$ we need to find, for each $i < 2m$, only the value of $h^*(i)$, then find, for each $i < 2m$, the $h^*(i)$ people that the person i knows, for then we shall have enough information on hand to determine whether there is a semi-solution for S_m^* extending the given semi-solution for S_{n+1}^*.

Define $h_1(i) = \mu t(\sum_{j<t} X_R(B(i), G(j)) = h^*(2i))$. Since h^* is recursive so is h_1, and every girl that $B(i)$ knows is among $G(0)$, $G(1)$, $\ldots$, $G(h_1(i))$.

Similarly, define $h_2(i) = \mu t(\sum_{j<t} X_R(B(j),\ G(i)) = h^*(2i+1))$. Then h_2 is recursive and every boy that $G(i)$ knows is among $B(0), B(1), \ldots, B(h_2(i))$. Hence $(s')_i$ will be at most $h_1(i)$ for each i and $(t')_i$ will be at most $h_2(i)$ for each i. Hence we may assume that s' is at most $\tilde{h}_1(m)$ and that t' is at most $\tilde{h}_2(m)$. Thus the predicate above can be written in the form

$$(\forall m)\,(\exists s')_{s'<\tilde{h}_1(m)}\,(\exists t')_{t'<\tilde{h}_2(m)}\,R^*(m, s', t', s, t, z, w, n),$$

which is in turn of the form

$$(\forall m)\,Q^*(m, s, t, z, w, n),$$

where Q^* is recursive.

We now define two functions f and g simultaneously by a course of values recursion:

$$f(n) = \mu z[(\exists w)_{w<h_2(n)}\,(\forall m)\,Q^*(m, \bar{f}(n), \bar{g}(n), z, w, n \dot{-} 1)],$$

$$g(n) = \mu w[(\forall m)\,Q^*(m, \bar{f}(n), \bar{g}(n), f(n), w, n \dot{-} 1)].$$

It is clear that these functions f and g are the ones referred to earlier. It is also clear that f is recursive in a predicate which is recursive in $\mathbf{0}'$. Hence the degree of f is $\leqslant \mathbf{0}'$.

Theorem 5* can be considerably strengthened, but again more powerful recursion-theoretic techniques are necessary; we leave both the statement and the proof for later (Theorem 11*).

We now return to the study of recursive societies which have a unique solution to their symmetric marriage problem. We note that the society constructed for Theorem 1* is an example of such a society, so we know immediately that there being a unique symmetric solution does not guarantee that that solution is recursive. We wish to discuss here what more can be said of such a society. The next two theorems, corresponding to Theorems 6 and 7 above, show that there is a recursive society with a unique symmetric solution and that solution has degree $\mathbf{d}$ if and only if $\mathbf{d} \leqslant \mathbf{0}'$.

THEOREM 6*. *Let S be a recursive society which has a unique solution to its symmetric marriage problem. Let $\mathbf{d}$ be the degree of that solution. Then* $\mathbf{d} \leqslant \mathbf{0}'$.

Proof. We remind the reader of the proof of Theorem 2*, where, with the additional hypothesis that S be highly* recursive, we proved that the unique symmetric solution must be recursive. It is easy to see that what was actually proved was that the unique solution is recursive in the function h^*, which there was recursive. Thus we need to show only that in general the function h^* is recursive in $\mathbf{0}'$.

(as by hair-style) but rather can be done in degree $\mathbf{d}$; in that case one obtains results which involve the degree $\mathbf{d}$ wherever the degree $\mathbf{0}$ is involved above.

5. In this paper we have focused on finding a solution, of as low a degree of unsolvability as possible, to the marriage problem of every solvable recursive society. In the original manuscript we raised the following two questions. Is it possible to strengthen Theorem 8 to complement Theorem 10 more fully by proving that if $\mathbf{d}' = \mathbf{0}''$ then there is a solvable recursive society with no solution recursive in $\mathbf{d}$? Is it possible to strengthen Theorem 3 to complement Theorem 11 more fully by proving that if $\mathbf{d}' = \mathbf{0}'$ then there is a highly recursive solvable society with no solution recursive in $\mathbf{d}$? Carl Jockusch has observed that both questions have negative answers.

Jockusch also made the following interesting conjectures. We appreciate his sharing them with us and his permission to report them here. If correct, the conjectures would establish the equivalence of many of the above results with known results concerning paths through recursive trees and retraceable sets. For definitions and further references, see **[6]** and **[14]**.

Conjecture 1. If T is a recursive finitely branching tree, then there is a recursive society S and a degree-preserving one-to-one correspondence between the set of infinite paths through T and the set of solutions to the marriage problem of S.

Conjecture 2. If T is a recursively bounded recursive tree, then there is a highly recursive society S and a degree-preserving one-to-one correspondence between the set of infinite paths through T and the set of solutions to the marriage problem of S.

It is an immediate observation that if S is a recursive society then there is in fact a recursive finitely branching tree T and a degree-preserving one-to-one correspondence between the set of infinite paths through T and the set of *all* solutions to the marriage problem of S. For example, we can let T be the set of sequence-numbers s such that a solution to the marriage problem of $S_{\mathrm{lh}\,(s)}$ (cf. the proof of Theorem 5) would consist of marrying $B(0)$ to $G(s_0)$, $B(1)$ to $G(s_1)$, ..., $B(\mathrm{lh}\,(s)-1)$ to $G(s_{\mathrm{lh}\,(s)-1})$. (The reader is cautioned that the enumeration of finite sequences given in § 1 is inappropriate here and that a slightly different, but still fully effective, encoding is implicitly being used here and throughout the remainder of this section.) If S is highly recursive, then T, as defined above, is recursively bounded, so that if S is a highly recursive society then there is a recursively bounded tree T such that S and T are related as in Conjecture 2. These observations make it possible for both conjectures to be correct.

Similarly, if S is a recursive society, then there is a recursive finitely branching tree T and a degree-preserving one-to-one correspondence between the set of infinite paths through T and the set of solutions to the *symmetric* marriage problem of S. For example, we can let T be the set of sequence-numbers encoding sequences $(\langle s_0, t_0\rangle, \langle s_1, t_1\rangle, \ldots, \langle s_n, t_n\rangle)$ such that $\langle s, t\rangle$ is a semi-solution for S_n^* in the sense of Theorem 5*. (Here s codes $\langle s_0, \ldots, s_n\rangle$ and t codes $\langle t_0, \ldots, t_n\rangle$.) Again, if S is highly recursive, then T, as defined above, is recursively bounded. These observations make it possible for the 'symmetric versions' of the conjectures to be correct.

We have been unable either to establish or to refute the conjectures. However, the three of us have verified that the 'symmetric versions' of the conjectures are correct.

THEOREM. *If T is an infinite recursive finitely branching tree, then there is a recursive society S such that there is a degree-preserving one-to-one correspondence between the set of infinite paths through T and the set of solutions to the symmetric marriage problem of S.*

Proof. Since the nodes of T form an infinite recursive set, we can recursively rename the set of boys using the sequence numbers in the tree above the root as nicknames. Similarly, we can rename the set of girls using this time all of the sequence numbers in T as nicknames. Each boy is to know only the girl whose nickname is identical with his and the girl whose nickname is the node immediately below his nickname. The boy with nickname s is denoted $♂s$ and the girl with nickname s is denoted $♀s$. For an example see Fig. 8. The degree-preserving one-to-one correspondence between solutions f to the symmetric marriage problem of S and infinite paths P through T is given by letting the path P correspond to the solution f if $P = \{s \,|\, f(♂s) \neq ♀s\} \cup \{\langle\,\rangle\}$.

If the recursive tree T in the above proof is assumed to be recursively bounded, then the society S defined above would be highly recursive. Thus, we have also proved that the symmetric version of Conjecture 2 is correct.

The conjectures have many consequences for the possible sets of solutions to the marriage problem of recursive societies. For example, if $1 \leqslant n \leqslant \aleph_0$, then there is a recursive society with solutions in exactly n

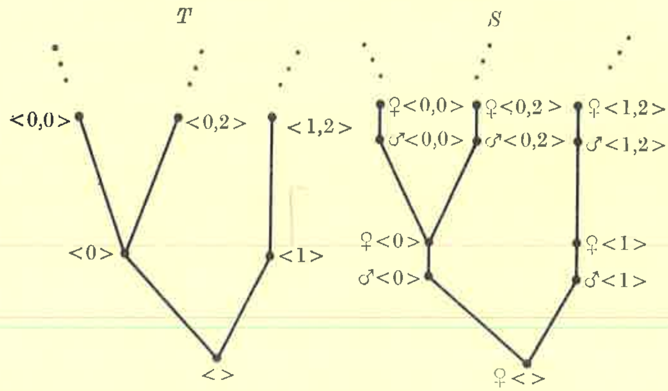

FIG. 8

degrees. We were able to establish this result directly, using a priority argument; its proof will appear elsewhere.

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