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TWO-DIMENSIONAL PARTIAL ORDERINGS: RECURSIVE MODEL THEORY

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In this paper and the companion paper [9] we describe a number of contrasts between the theory of linear orderings and the theory of two-dimensional partial orderings.

The notion of dimensionality for partial orderings was introduced by Dushnik and Miller [3], who defined a partial ordering $\langle A, R \rangle$ to be n -dimensional if there are n linear orderings of A , $\langle A, L_1 \rangle$, $\langle A, L_2 \rangle$, ..., $\langle A, L_n \rangle$ such that $R = L_1 \cap L_2 \cap \dots \cap L_n$. Thus, for example, if Q is the linear ordering of the rationals, then the (rational) plane $Q \times Q$ with the product ordering $\langle \langle x_1, y_1 \rangle \leq_{Q \times Q} \langle x_2, y_2 \rangle$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2 \rangle$ is 2-dimensional, since $\leq_{Q \times Q}$ is the intersection of the two lexicographic orderings of $Q \times Q$. In fact, as shown by Dushnik and Miller, a countable partial ordering is n -dimensional if and only if it can be embedded as a subordering of Q^n .

Two-dimensional partial orderings have attracted the attention of a number of combinatorialists in recent years. A basis result recently obtained, independently, by Kelly [7] and Trotter and Moore [10], describes *explicitly* a collection \mathcal{P} of finite partial orderings such that a partial ordering is a 2dpo if and only if it contains no element of \mathcal{P} as a subordering. The existence of such a \mathcal{P} was known earlier, and can be proved easily using a compactness argument. (Alternatively, see the review of Harzheim [4].) Baker, Fishburn, and Roberts [1] showed that the theory of 2dpo's is not finitely axiomatizable, although it follows from the existence of \mathcal{P} above that it is axiomatized by universal statements.

In this paper, we shall see that, at least from the perspective of recursive model theory, the class of 2dpo's behaves more like the class of partial orderings than like its one-dimensional subclass, the class of linear orderings. For each of the following sample negative results about 2dpo's, the positive version is true for linear orderings.

- (1) There is a recursive partial ordering which is a 2dpo but there is no recursive function which embeds it in $Q \times Q$.
- (2) There is a recursive partial ordering which is a 2dpo but is not isomorphic to any recursive subordering of $Q \times Q$.
- (3) There is a recursively enumerable 2dpo which is not isomorphic to any recursive 2dpo.

These results grew out of our attempt to understand the effective content of the following analogous facts.

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I. Every countable linear ordering is order-isomorphic to a subset of Q ;

II. Every countable 2dpo is order-isomorphic to a subset of $Q \times Q$.

Suppose we have a recursive ordering. Will there be a recursive function which maps it onto a recursive subset of Q if it is a linear ordering? Will there be a recursive function which maps it onto a recursive subset of $Q \times Q$ if it is a 2dpo? Alternatively, how effective is Cantor's proof of I and its generalization for II?

We will first see that the proof of I is completely effective. Before doing so, however, we define the notion of a recursive ordering. Since a partial ordering is a binary relation R on a set A , it seems appropriate to define a partial ordering to be a *recursive ordering* if R is a recursive relation on a recursive set A . (This of course presupposes that A is identified with some subset of the natural numbers N . Note also that requiring A to be recursive is unnecessary: for, since R is reflexive, we can tell whether or not $a \in A$ by determining whether or not $\langle a, a \rangle \in R$.)

Since we will be mainly concerned with partial orderings we will henceforth denote the binary relation R on A by \leq_A and we will often abuse our terminology by referring to $\langle A, \leq_A \rangle$ simply as A ; conversely, when we speak of an ordering A we imply that we are speaking of a particular binary relation on A , which has been suppressed from the notation exclusively for typographical reasons. As usual, we will write $a <_A b$ to mean that $a \leq_A b$ but that $b \not\leq_A a$. We also use the symbol $|$ to represent incomparability.

In this paper, we interpret Q as a recursive ordering by identifying the rationals in some canonical way with the natural numbers, so that the associated enumeration of Q is r_0, r_1, r_2, \dots and the ordering of Q is identified with the recursive set $\{\langle i, j \rangle \mid r_i \leq_Q r_j\}$. Similarly, we interpret $Q \times Q$ as a recursive ordering by identifying, in some canonical way, the elements of $Q \times Q$ and N ; the associated enumeration is then $Q \times Q = \{\langle x_i, y_i \rangle \mid i \in N\}$ and the ordering of $Q \times Q$ is then identified with the recursive set $\{\langle i, j \rangle \mid \langle x_i, y_i \rangle \leq_{Q \times Q} \langle x_j, y_j \rangle\}$.

One further convention: Whenever we speak of a recursive ordering M , we always tacitly assume that M is identified in some fixed way with a subset of N . Thus when we say "choose the first element of M such that \mathcal{P} " we will always mean "of the various elements of M which satisfy \mathcal{P} , choose the one which corresponds to the smallest element of N ".

Theorem 1 is the kind of fact that exists in the folklore long before it is first uttered.

THEOREM 1. *Let M be a recursive linear ordering. Then there is a recursive function which maps it isomorphically onto a recursive subset of Q .*

PROOF. Repeat verbatim the classical construction which embeds a countable linear ordering M into Q by enumerating $M = \{m_0, m_1, m_2, \dots\}$ and by defining the map $f: M \rightarrow Q$ inductively. This map becomes recursive if M is enumerated recursively and if, in defining each $f(m_i)$, we specify that it shall be the first appropriate member of Q .

The range of f may not be recursive, although it is recursively enumerable, because one can never tell whether or not an element of Q , which is not yet in the range of f , will eventually be in its range. To circumvent this problem, we need only modify the construction above so that we choose for $f(m_i)$ the first appropriate element of Q which is not among the finitely many elements $r_0, r_1, r_2, \dots, r_i$ of Q .

Then, r_k will be in the range of f if and only if $r_k = f(m_i)$ for some $i \leq k$ and the range of f will be recursive. ■

We observe that, as an alternative to modifying the construction in the proof above, we could have deduced the desired result from the following observation, whose proof is similar. Given any recursively enumerable subset A of Q (such as the range of the original f above), there is a recursive subset B of Q with the same order type such that an isomorphism between the two can be effected by a recursive map.

Moreover, we phrased the proof so that it depended only on M being recursively enumerable. Thus if we define a partial ordering $\langle M, \leq_M \rangle$ to be recursively enumerable (RE) if both M and \leq_M are RE, then any RE linear ordering satisfies the conclusion of Theorem 1. (The experienced uniformist will observe that there are recursive functions which, given an index for M , will produce indices for the recursive subset of Q and for the recursive isomorphism.)

As a consequence of Theorem 1, we can see that a variety of different possible definitions of recursive linear order types are equivalent.

COROLLARY 2. *Let τ be an infinite linear order type. Then the following are equivalent.*

- (1) *There is an RE linear ordering of order type τ .*
- (2) *There is an RE linear ordering of order type τ whose field is N .*
- (3) *There is a recursive linear ordering of order type τ .*
- (4) *There is a recursive linear ordering of order type τ whose field is N .*
- (5) *There is an RE subset of Q which has order type τ .*
- (6) *There is a recursive subset of Q of order type τ .*
- (7) *There is a recursive linear ordering whose field is N and a recursive function mapping it isomorphically onto a subset of Q of order type τ .*

PROOF. It suffices to present a proof that (1) implies (4). That, together with Theorem 1 and the subsequent remarks, yields all of the other implications. So assume that $\langle M, \leq_M \rangle$ is an RE linear ordering. Enumerate the infinite set M in some fixed effective manner, assigning to the i th element m_i enumerated in M the number i . When $m_i \leq_M m_j$ is enumerated in \leq_M , place $\langle i, j \rangle$ into R . Then $\langle N, R \rangle$ is clearly an RE linear ordering isomorphic to $\langle M, \leq_M \rangle$. Furthermore, since for each i and j either $m_i \leq_M m_j$ or $m_j \leq_M m_i$, we conclude that either $\langle i, j \rangle \in R$ or $\langle j, i \rangle \in R$ (but not both unless $i = j$) and hence that R is actually recursive. ■

We define an order type τ to be a *recursive order type* if it satisfies any one, and hence all, of the above conditions.

We turn now to the 2dpo case and show that the situation is quite different. Let us define M to be *recursively embeddable in the plane* if it is recursively isomorphic to a subset of $Q \times Q$. (That is, there is a recursive function f with domain M which satisfies $a \leq_M b$ iff $f(a) \leq_{Q \times Q} f(b)$.) We say that M is *embeddable as a recursive subset of the plane* if there is a function f , whose domain is M and whose range is a recursive subset of $Q \times Q$ satisfying $a \leq_M b$ iff $f(a) \leq f(b)$. If, in addition, f

THEOREM 3. *There is a recursive 2dpo which is not recursively embeddable in the plane, and hence is certainly not recursively embeddable as a recursive subset of the plane.*

PROOF. We shall construct a recursive partial ordering (B, \leq_B) of N which is composed of countably many disjoint sets denoted B_0, B_1, B_2, \dots . We refer to the set B_i as the i th box and agree that $x \in B_i, y \in B_j$, and $i < j$ imply $x <_B y$. The construction of $<_B$ is described as an enumeration of $<_B$ in stages with the relations just specified understood implicitly.

The e th box will contain three pairwise incomparable elements, denoted a_0^e, a_1^e and a_2^e , and may also contain one additional element, denoted b_e . If it is defined, b_e will ensure that φ_e is not a recursive embedding of \leq_B into the plane.

We now describe the construction. There are two types of stages depending upon whether the stage is or is not a power of 2.

Stage 2^e . Define a_0^e, a_1^e, a_2^e to be the three least elements of N not yet used in the construction.

Stage $2^e(2s + 1)$ [$s \geq 1$]. Go directly to the next stage of the construction unless all of the following conditions are met.

- (1) b_e has not been defined at an earlier stage of the construction;
- (2) $\varphi_e^s(a_i^e)$ is defined for each $i \leq 2$;
- (3) The three elements $\varphi_e(a_i^e), i \leq 2$, are pairwise incomparable in $Q \times Q$.

Let t be that permutation of $\{0, 1, 2\}$ for which the first components of $\varphi_e(a_{i(0)}^e), \varphi_e(a_{i(1)}^e), \varphi_e(a_{i(2)}^e)$ are in increasing order. Our third assumption shows that the second components of this sequence must be in decreasing order.

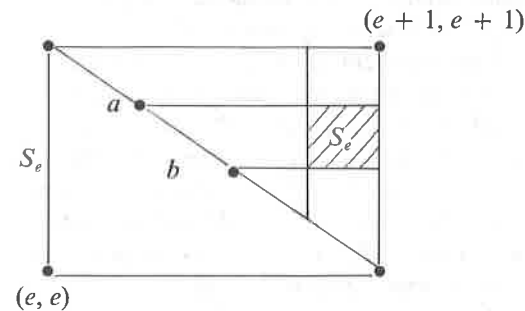
$$\begin{aligned} &\varphi_e(a_{i(0)}^e) \cdot \\ &\quad \varphi_e(a_{i(1)}^e) \cdot \\ &\quad \varphi_e(a_{i(2)}^e) \cdot \end{aligned}$$

Let b_e be the first element not yet used in the construction. Enumerate $b_e >_B a_{i(0)}^e$ and $b_e >_B a_{i(2)}^e$.

This completes the description of the construction. $<_B$ is recursive since, for any elements x and y of N , as soon as both are used in the construction, the $<_B$ relation between them (including incomparability) is established. In particular, if b_e is defined, $b_e | a_{i(1)}^e$ so that φ_e cannot be an embedding of $<_B$ into $Q \times Q$. Finally, we note for completeness that $<_B$ is easily embedded in the plane although not recursively. ■

Theorem 3 is similar in flavor and proof to several other known theorems concerning recursive aspects of combinatorial problems. We cite three examples. There is a recursive tree which has an infinite path but no recursive infinite path. (See [6] for a complete analysis.) There is a recursive society which has a solution to its marriage problem but no recursive solution [8]. There is a recursive partition of $N^{(2)}$ for which there is a homogeneous set but no recursive homogeneous

To construct a recursive subset T of $Q \times Q$ which is isomorphic to B , we first exclude from T all points of $Q \times Q$ not within one of the squares S_e :



Place in T three uniformly chosen diagonal points of S_e , say a , b and c , and exclude from T all other points of S_e except those in S'_e . For each e and t , see whether b_e is newly defined at stage t of the construction of $<_B$. If so, put into T the first element of $Q \times Q$ which is in S'_e and is not among the first t elements of $Q \times Q$. (Thus, for any s , if the s th element of $Q \times Q$ is not in T by time s it will never be in T .) It is then clear that T is order-isomorphic to B and is a recursive subset of the plane.

We will next construct a recursive 2dpo which is not even embeddable as a recursive subset of the plane. The proof makes use of the following combinatorial lemma, which appears interesting in its own right. We do not know whether this result was previously known, nor whether our example is the simplest possible.

LEMMA 4. *There is an infinite sequence of finite 2dpo's, no one of which is embeddable in any of the others.*

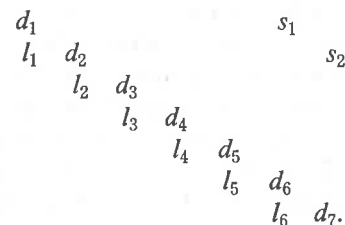
PROOF. For $n \geq 4$, let D_n be the partial ordering with $2n + 1$ points $d_1, \dots, d_n, l_1, \dots, l_{n-1}, s_1$, and s_2 , ordered by the transitive closure of

$$l_i < d_i \text{ and } l_i < d_{i+1} \text{ for } 1 \leq i < n,$$

$$d_i < s_1 \text{ for } 1 \leq i < n, \text{ and}$$

$$d_j < s_2 \text{ for } 1 < j \leq n.$$

For example, D_7 is represented graphically by



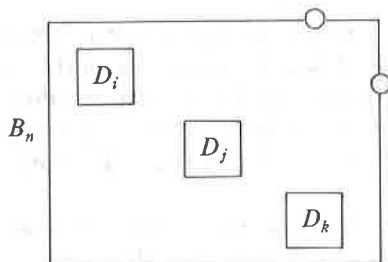
The points d_i are called diagonal points of D_n , the points l_j subdiagonal, and s_1 and s_2 superdiagonal. When it is necessary to distinguish points in D_n from those

chains of length 3 are $l_j < d_i < s_k$ for some j, i, k . This implies that φ must carry diagonal points to diagonal points, subdiagonal points to subdiagonal points, and superdiagonal points to superdiagonal points.

Since $\varphi(s_1^n) \in \{s_1^m, s_2^m\}$, we shall assume $\varphi(s_1^n) = s_1^m$. This assumption simplifies notation, but a dual argument can be carried out in case $\varphi(s_1^n) = s_2^m$. Since d_1^n is the only diagonal point incomparable with s_2^n in D_n and d_1^m is the only diagonal point incomparable with s_2^m in D_m , $\varphi(d_1^n) = d_1^m$. Since l_1 is the only subdiagonal point less than d_1 in both D_n and D_m , $\varphi(l_1) = l_1$. Since d_2 is the only diagonal point greater than l_1 other than d_1 in both D_n and D_m , $\varphi(d_2) = d_2$. Since l_2 is the only subdiagonal point less than d_2 other than l_1 in both D_n and D_m , $\varphi(l_2) = l_2$. Continuing, we see that $\varphi(d_n^n) = d_n^m$. Since $d_n^n | s_1^n$, $\varphi(d_n^n) = d_n^m | s_1^m = \varphi(s_1^n)$. This can only happen if $m = n$ so that the lemma is proved. ■

THEOREM 5. *There is a recursive partial ordering which is embeddable in the plane, but is not embeddable as a recursive subset of the plane.*

PROOF. We shall construct a recursive partial ordering (R, \leq_R) of N which is composed of a sequence of boxes B_0, B_1, B_2, \dots satisfying the conditions described in the first paragraph of the proof of Theorem 3. Each box B_n will contain one copy of each of three of the sets in the sequence $\{D_m | m \geq 4\}$ and possibly, two additional points. A two-dimensional representation of B_n would look like



where the two circled points may be absent.

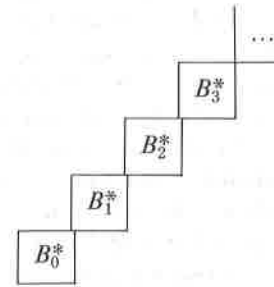
The construction will proceed during the simultaneous enumeration of two recursively enumerable sets, W and V , which are recursively inseparable. If n turns up in W , two points will be added to B_n so that any two-dimensional representation of B_n will have a certain form; while if n turns up in V , they will be added to B_n so that any two-dimensional representation of B_n will have a different form. If all this is done carefully, any recursive subset of the plane isomorphic to R would yield a recursive separation of W and V , contrary to hypothesis.

We proceed to the construction. Using all of the even natural numbers we generate recursively one example of each of the partial orderings $\{D_n | n \geq 4\}$. We place the points of $D_{3n'+i}$ into B_n for $i = 0, 1, 2$; here, and subsequently, n' is $n + 1$. The only other comparability relations among the even numbers are determined by our convention on the partial ordering to the boxes B_0, B_1, B_2, \dots .

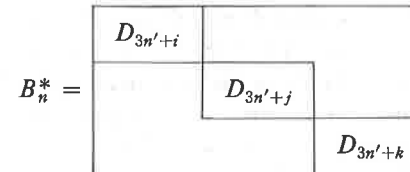
We now simultaneously enumerated W and V in a 1-1 manner. When n is enumerated in either W or V , the first two odd numbers not yet used are placed in the box B_n . If n is enumerated in W , the first of the two points is placed above all points of $D_{3n'+1}$ and $D_{3n'+2}$ but incomparable with all points of $D_{3n'+3}$, while the second is placed above all points of $D_{3n'+1}$ and $D_{3n'+3}$ but incomparable with all points of

$D_{3n'+2}$. If, on the other hand, n is enumerated in V , the first of the two points is placed above all points of $D_{3n'+2}$ and $D_{3n'+3}$ but incomparable with all points of $D_{3n'+1}$, while the second is placed above all points of $D_{3n'+1}$ and $D_{3n'+3}$ but incomparable with all points of $D_{3n'+2}$.

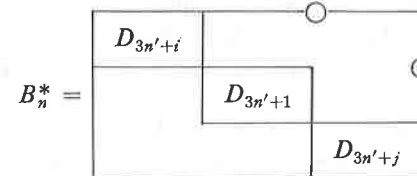
This completes the construction. It is evident that R is recursive and can be embedded in the plane. Indeed, a subset B^* of the plane has the same order type as R if and only if it is decomposable into



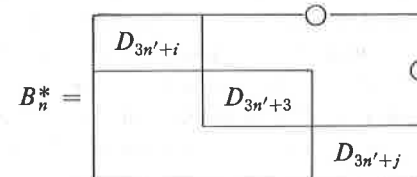
where for each n if $n \notin W \cup V$ then



for some permutation $\{i, j, k\}$ of $\{1, 2, 3\}$; if $n \in W$ then



for some permutation $\{i, j\}$ of $\{2, 3\}$; and, if $n \in V$ then



for some permutation $\{i, j\}$ of $\{1, 2\}$.

B_m^* , and since there is an element of E below each pair of adjacent diagonal elements of E , it is readily verified that all the diagonal elements of E must lie in the same $D_{3m'+j}$. Since subdiagonal and superdiagonal elements of E are always comparable with some diagonal elements and incomparable with others, we see that they must all be within the same $D_{3m'+j}$. By Lemma 4, E must actually be $D_{3m'+j}$. Thus if $t = 3n' + i$, then $n' = m'$ and $i = j$ and E is the subset of B_n^* corresponding to D_t , as claimed.

Suppose now that B^* is a recursive subset of the plane with the same order type as R . We then obtain a recursive set F which separates V and W as follows. Let n be given. Search B^* until subsets $S_{3n'+1}$, $S_{3n'+2}$, $S_{3n'+3}$ of order types $D_{3n'+1}$, $D_{3n'+2}$, $D_{3n'+3}$ are found. (This search must succeed.) We know that the sets found make up the bulk of B_n^* . Let p_1 , p_2 and p_3 be points of $S_{3n'+1}$, $S_{3n'+2}$ and $S_{3n'+3}$ respectively. Determine which of the three is left-upper-most and which is right-lower-most, and let p_t be the remaining one. If $t = 1$, put $n \in F$; otherwise, $n \notin F$. Clearly F is recursive and $W \subseteq F$; also if $n \in V$, p_t must be p_3 , so that $n \notin F$. Thus F determines a recursive separation of W and V , contrary to hypothesis.

Hence R is a recursive partial ordering which is embeddable in the plane, but is not embeddable as a recursive subset of the plane. ■

As a consequence of Theorems 3 and 5, the 2dpo version of Corollary 2 is somewhat different. Indeed, the various possible definitions of recursive 2dpo types are inequivalent. For each m , $1 \leq m \leq 7$, let T_m be the set of infinite 2dpo types satisfying condition (m) of Corollary 2 (where each occurrence of "linear ordering" is replaced by "2dpo" and each occurrence of " Q " is replaced by " $Q \times Q$ ").

THEOREM 6. $T_7 = T_6 = T_5 \subsetneq T_4 = T_3 \subsetneq T_2 = T_1$.

All of the inclusions $T_{i+1} \subseteq T_i$ are clear except possibly $T_5 \subseteq T_4$ which follows from an argument similar to, and slightly easier than, our proof in Corollary 2 that (1) implies (4). This type of argument may also be used to show that $T_1 \subseteq T_2$, $T_3 \subseteq T_4$ and $T_6 \subseteq T_7$. That $T_5 \subseteq T_6$ will be stated and proved as Theorem 6A. Theorem 5 shows that $T_5 \neq T_4$. An example to show that $T_3 \neq T_2$ will be provided in Theorem 6B.

THEOREM 6A. Let A be an RE subset of $Q \times Q$. Then there is a recursive subset B of $Q \times Q$ such that $B \simeq A$. Moreover, B can be chosen so that if (x_1, y_1) , (x_2, y_2) are different points of B , then both $x_1 \neq x_2$ and $y_1 \neq y_2$.

PROOF. Let φ be an isomorphism from a subset C of $Q \times Q$ into $Q \times Q$. We say that φ is a special isomorphism if whenever $p_1 <_C p_2$ and $\varphi(p_i) = (x_i, y_i)$ for $i = 1, 2$, then $x_1 < x_2$ and $y_1 < y_2$. Notice that the image of any subset of $Q \times Q$ under a special isomorphism automatically satisfies the last statement of the theorem.

The density of Q shows the correctness of the following assertion, which is a special case of the "Duplication Lemma" of Crossley and Nerode [2, p. 24].

Assertion. Let φ be a special isomorphism defined on a finite set C . Let E_1 and E_2 be finite subsets of Q and suppose that $x \notin C$. Then there is a point $u = (u_1, u_2)$

At stage $s + 1$, extend φ to include a_{s+1} in its domain using the assertion with $E_1 = E_2 = \{r_0, \dots, r_s\}$. The recursiveness of $B = \varphi(A)$ follows easily since $(r_i, r_j) \in B$ iff $(r_i, r_j) \in \{\varphi(a_0), \dots, \varphi(a_{\min(i,j)})\}$. ■

The final statement of the theorem will be used soon in effectivizing a proof of Dushnik and Miller.

THEOREM 6B. *There is an RE 2dpo which is not isomorphic to any recursive 2dpo.*

PROOF. We can enumerate a partial ordering (B, \leq_B) of N which is composed of a sequence of boxes B_0, B_1, B_2, \dots satisfying the conditions described in the first paragraph of the proof of Theorem 3. Each box B_n will contain two copies of the 2dpo D_{n+4} defined in the proof of Lemma 4. Let K be an RE set which is not recursive. If $n \notin K$, each point in one copy of D_{n+4} will be incomparable with each point in the other. If $n \in K$, all the points in one of the copies of D_{n+4} will be greater than all of the points in the other copy of D_{n+4} . Finally, an argument like that at the end of the proof of Theorem 5 shows that if (C, \leq_C) were recursive and isomorphic to (B, \leq_B) , then K would be recursive. ■

Let us return now to the questions we posed earlier. We found that although I has a natural effectivization for recursive linear orderings, no reasonable effectivization of II is correct for recursive 2dpo's. One explanation for this is that in defining "recursive orderings" we just simply mimicked the definition of "recursive linear orderings"—not an unreasonable procedure. However, as is apparent from the outcome, a recursive ordering which happens to be a linear ordering is a more recursive object than a recursive ordering which happens to be a 2dpo; thus the failure of II to effectivize properly can be attributed to the insistence on defining "recursive linear ordering" and "recursive 2dpo" in precisely the same way. Had we not insisted on giving uniform definitions, we might have defined a partial ordering $\langle A, R \rangle$ to be recursive 2dpo if there are two recursive linear orderings $\langle A, S \rangle$ and $\langle A, T \rangle$ so that $R = S \cap T$. (Since this is the second version of a definition and since it is stronger, we distinguish it from the first by a superscript 2.)

We first observe that every recursive subset of $Q \times Q$ has the same order type as a recursive² 2dpo (so that this definition is not overly restrictive). Let $A \subseteq Q \times Q$; as in the proof of Theorem 6A, we may assume that any two points of A have different x -coordinates and y -coordinates. Given $a \in Q \times Q$, we write $a = \langle x_a, y_a \rangle$. Define $\langle A, S \rangle$ by specifying that aSb iff $x_a \leq x_b$ and define $\langle A, T \rangle$ by specifying that aTB iff $y_a \leq y_b$. Then $\langle A, S \rangle$ and $\langle A, T \rangle$ are recursive linear orderings whose intersection is $\langle A, \leq_{Q \times Q} \rangle$.

The following converse, which provides an effectivization of II, is based on Dushnik and Miller's proof of II.

THEOREM 2. *Let $\langle A, R \rangle$ be a recursive² 2dpo. Then there is a partial recursive function which maps it isomorphically onto a recursive subset of $Q \times Q$.*

PROOF. Let $R = S \cap T$ where $\langle A, S \rangle$ and $\langle A, T \rangle$ are recursive linear orderings. Let f_S and f_T be partial recursive functions which map $\langle A, S \rangle$ and $\langle A, T \rangle$ respectively onto recursive subsets of O . Define $f: A \rightarrow O \times O$ by $f(a) = (f_S(a), f_T(a))$.

so that f is an isomorphism from A into $Q \times Q$. Clearly f is partial recursive and the image of f is a recursive subset of $Q \times Q$. ■

Perhaps then the new definition of recursive² 2dpo is more appropriate than the first definition of recursive 2dpo even though the first is "uniform" and the second depends on the class of models. Another way of arriving at the same conclusion is by looking at the effective version of the notion of universal models. Classically, a structure is universal for a class of similar structures if each structure in the class can be embedded in the given structure. Thus Q is universal for the class of countable linear orderings and $Q \times Q$ is universal for the class of countable 2dpo's. Theorem 1 shows that Q is recursively universal for the class of recursive linear orderings in the strongest possible sense; that is, each recursive linear ordering can be recursively embedded as a recursive subset of Q . On the other hand, Theorems 3 and 5 show that $Q \times Q$ is not recursively universal for the class of recursive 2dpo's in any reasonable sense.

Although it is possible to construct a recursively universal model for the class of recursive 2dpo's (Theorem 8), such a model is, from an intuitive point of view, hardly a satisfactory substitute for $Q \times Q$. Its order type cannot be the same as that of $Q \times Q$, since the only property of $Q \times Q$ used in Theorem 5 was the following fact which is also true in any isomorphic copy of $Q \times Q$: given any three pairwise incomparable points, for exactly one of them it is true that anything bigger than both of the others is also bigger than it. Furthermore, although the theory of $Q \times Q$, with constants for elements of $Q \times Q$, is decidable, it can be shown, using the partial ordering constructed in Theorem 5, that if U is a recursive 2dpo which is recursively universal for the class of recursive 2dpo's, then even the existential statements of the theory of U , with constants for elements of U , is undecidable (Theorem 9).

THEOREM 8. *There is a recursively universal 2dpo. That is, there is a recursive 2dpo U such that if $\langle A, R \rangle$ is a recursive 2dpo there is a partial recursive isomorphism of A onto a recursive subset of U .*

PROOF. Note that a countable partial ordering is a 2dpo iff every finite subordering is a 2dpo, and that there is an effective method for determining whether or not a finite binary relation is a 2dpo. The 2dpo $U = \langle N, R \rangle$ will consist of a sequence of boxes $B_0, B_1, B_2 \dots$ as described in the first paragraph of the proof of Theorem 3; now, however, each box will contain infinitely many points. In fact, the e th box B_e will consist of all the numbers $\{b_i^e \mid i \in N\}$ of the e th set of a recursive partition of N into infinitely many recursive sets. (We assume that b_{ei} is a recursive function of e and i and that for each e , $\{b_i^e \mid i \in N\}$ is enumerated in increasing order.) The effect of the construction is to make B_e look like

	R^e

where L_e is a linear ordering of order type ω and R^e is either a finite 2dpo or, if φ_e is the characteristic function of a recursive 2dpo in T_k , R^e is isomorphic to that 2dpo by a partial recursive function. We omit the details of the construction, giving only a brief description of the pertinent features. At each stage of the construction, R^e will be isomorphic to the 2dpo defined by φ_e^s on the largest possible initial segment of N . In the meantime, we expand L^e upwards, being certain to place b_i^e in B_e at the i th stage of the construction if it has not yet been placed. We leave the verification that this can be done, and that the resulting U has the desired properties, to the reader. ■

We can again use Theorem 5 to strengthen our earlier observation that U cannot be classically isomorphic to $Q \times Q$. In the final theorem below, we show that U cannot be recursively presented in the sense of [2].

THEOREM 9. *Let U be a recursively universal recursive 2dpo. The complete diagram of U , i.e., the theory of the expansion of U by adjoining constants c_n interpreted as n for each $n \in U$, is undecidable. Indeed, the set of existential sentences of this theory is undecidable.*

PROOF. Let \mathcal{B} be the recursive 2dpo constructed in the proof of Theorem 5 and let W and V be the disjoint recursively inseparable recursively enumerable sets used in the construction of \mathcal{B} . Let \mathcal{B}^* be a recursive subordering of U and h a recursive isomorphism of \mathcal{B} onto \mathcal{B}^* . Let p_1^n, p_2^n and p_3^n be recursive functions of n such that for each $n \in N$ and $i \in \{1, 2, 3\}$, $p_i^n \in D_{3n'+i} \subseteq B_n$ in \mathcal{B} . The construction of \mathcal{B} shows that such functions exist. Let S be the set of those n for which

$$(\exists x)(h(p_1^n) < x \wedge h(p_2^n) < x \wedge h(p_3^n) \not< x)$$

is true in U . $W \subseteq S$ while $V \cap S = \emptyset$. Thus S is not recursive and the set of existential sentences in the complete diagram of U is not decidable. ■

What conclusions can we draw from these results? For combinatorial theory, it suggests that one definition of 2dpo (intersection of two linear orderings) has lower combinatorial complexity than another (embeddability in the plane). For recursive model theory, it suggests that there really is no one notion of "recursive model" which will work best in all situations and that, for different classes of structures, different definitions of recursive model might be appropriate.

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