Chapter 1: Coloring Mathematically

Section 1.4. Using Vertex Coloring to Resolve Conflicts

Projects	Project Groups
Dinosaurs	Sarita, Barbara, Ravi
Spain	Sarita, Roberto
Bicycles	Roberto, Maimuna
Muscles	Maimuna, Boris, Christie
Fairy Tales	Barbara, Boris, Jason
Hockey	Ravi, Christie, Jason

Figure 65

In Section 1.3, we saw how graph coloring can be used to color maps and pictures. In this section, we will see a few of the many other applications of graph coloring.

Imagine that eight students in your class are working on six projects; the table in Figure 65 lists both the projects and the students who are working on each project.

You want to arrange meeting times after school for the project groups to meet. Each project group will need to meet after school on Monday, Tuesday, Wednesday, Thursday, or Friday, and, of course, you want to stay after school as few days as necessary.

However, since some of the students are working on more than one project, you need to be careful not to schedule those projects at the same time. For example, Sarita is working on the Dinosaurs project and the Spain project, so you can't schedule both these projects on the same day since this would create a dilemma for Sarita.

Can you help them out? How many days will be needed for the six projects?

One possible way of solving this problem is by looking at each project in turn and assigning its group to meet on a specific day. Thus you can start by assigning Monday to the Dinosaurs group. Next you consider the Spain group. It can't meet on Monday since Sarita, as noted above, is in both the Dinosaurs group and the Spain group. So the Spain group must meet on a different day, let's say Tuesday.

What about the Bicycles group? It can't meet on Tuesday because Roberto is in both the Spain group and the Bicycles group. But the Bicycles group can meet on the same day as the Dinosaurs group, since the two groups have no student in common. Of course, it doesn't have to meet on Monday; it could meet instead on Wednesday, Thursday, or Friday.

Just reading the last paragraph aloud should make you realize that assigning days to projects is like assigning colors to states.

- You have to use different colors if two states share a border.
- You have to use different days if two project groups share a student.

Moreover,

- You want to use the smallest number of colors for your maps.
- You want to use the smallest number of days for your projects.

Chapter 1: Coloring Mathematically

Section 1.7. Mathematical Reasoning

What are proofs and why are they needed?

In mathematics, if we assert that a statement is true, we need to verify that it really is true. Claims that have not been verified cannot be accepted as true.

An example of a famous claim which has not been verified, although there is tons of evidence to support it, is given in "Goldbach's Conjecture" that follows these paragraphs.

How does one verify that a statement is true? Mathematicians require a "proof", that is, **convincing reasoning that the statement is indeed true**.

It is possible to give a precise definition of what constitutes a "proof", and to write computer programs that check to see whether someone's "proof" is really a "proof."

However, writing proofs that can be examined by a computer program is so tedious that essentially no mathematicians do that.

Instead they rely on their own and their colleagues' evaluation of their "proofs" to determine if what they have presented is indeed convincing reasoning for their claims.

In this book, each time you find a claim, make sure that you understand the reasoning behind the claim and, if you are not convince by that reasoning, you should review that reasoning with another student or with a teacher to determine whether the reasoning is correct (and you are convinced by it) or not (in which case you should inform the author).

Goldbach's Conjecture

One famous claim that has not been verified is that every even number that is 6 or more is the sum of two odd primes. For example, $100 = 47 + 53$, $102 = 5 + 97$, $104 = 7 + 97$, and $106 = 53 + 53$.

This claim has been checked by computers for all even numbers up to $4 \cdot 10^{18}$ (that is, 400,000,000,000,000,000), and each of those even numbers is in fact the sum of two odd primes.

But it might be that some even number with 100 digits is not the sum of two odd primes.

"Every chain has chromatic number 2" is an example of what is called **an "if ... then ..." statement**, because what it says is "**if** a graph is a chain, **then** the graph has chromatic number 2."

We encounter "if ... then ..." statements in daily life. Some examples are:

- **If** it rains, **then** I won't go to the beach.
- **If** I study, **then** I will do well on the exam.
- **If** I eat another slice of meat loaf, **then** I won't have room for dessert.
- **If** you give me a kiss, **then** I'll go to sleep.

We sometimes use "if ... then ..." statements that are slightly disguised. Here are a few examples:

- I'll go to sleep if you give me a kiss.
- I'll only go to sleep if you give me a kiss.
- I won't go to sleep unless you give me a kiss.

The first statement says that "if you give me a kiss, then I'll go to sleep," although the order of the phrases is reversed.

The second statement says that "if I go to sleep, then [it will be because] you give me a kiss" but doesn't guarantee that if I get a kiss then I will go to sleep. It sounds more like "'if you don't give me a kiss, then I won't go to sleep."

The third statement has essentially the same meaning as the second statement – no deal has been made.

Actually, the statement "if you give me a kssl, then I'll go to sleep" sounds like a deal, because it says implicitly that "if you don't, I won't." We'll return to this question later in the section.

"If ... then ..." statements play an important role in every mathematical topic. In geometry, for example, "SAS" is an abbreviation for the statement "**if** two sides of one triangle and the angle they enclose have the same measures as two sides of another triangle and the angle they enclose, **then** the two triangles are congruent."

In algebra, when we multiply both sides of an equation by a fixed number k to get another equation, we use the fact that "**if** two quantities A and B are equal, **then** kA and kB are also equal."

A more concise way of expressing this fact is "$A = B$ implies $kA = kB$." Thus an "if ... then ... " statement is often called an **implication**.

We have come across several "if ... then ..." statements in this chapter:

Biographical Note

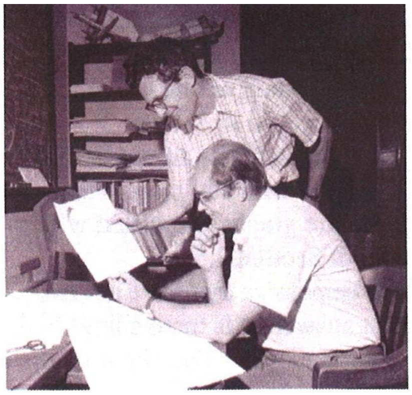

FIGURE 123

Kenneth Appel (at left) was born in 1932 in Brooklyn, NY. He graduated from Queens College in 1953, served in the army for two years and then got his PhD at the University of Michigan in 1959. He joined the faculty of the University of Illinois in 1961, retiring in 1993. He then spent ten years as chair of the Mathematics Department at the University of New Hampshire.

Wolfgang Haken (at right) was born in 1928. He received his bachelors and PhD degrees from University of Kiel, Germany. He solved the Triviality Problem for Knots, a major open problem, in time for the result to be announced at the 1954 International Congress of Mathematicians. He permanently joined the Illinois faculty in 1964 (after a visiting year in 1962).

In 1976, when they finished their work on the problem, they asked their children to help check for errors. Armin Haken, then a college freshman, Dorothea Haken and Andrew Appel, then high school seniors, and Laurel Appel, then a high school sophomore all joined in on the fun!

"Four Colors Suffice"

"Four Colors Suffice: How the Map Problem Was Solved" is the title of a very interesting and readable book by Robin Wilson about the history and the proof of the Four Color Theorem. It was published by the Princeton University Press in 2002.

How Mathematics Progresses

This brief historical account of the Four Color Conjecture and Four Color Theorem in 1976 by Kenneth Appel and Wolfgang Haken. (See Figure 123 and the "Biographical Note" in the side column.) In their proof, they showed that it was sufficient to verify that several thousand specific maps could be colored using four colors, and then had a computer check all those examples. Their result was newsworthy both because they solved an important unsolved problem in mathematics and because of their computer-assisted methodology; Figure 124 shows how the proof of the Four Color Conjecture was commemorated by the United States Postal Service in 1976.

FIGURE 124

Since 1976 the statement previously referred to as the "Four Color Conjecture" has been known as the "Four Color Theorem". A reflection on this historical account is presented in the side column (see "How Mathematics Progresses"), as is a description of a very interesting and readable book, "Four Colors Suffice," about the history of the map problem.

> **The Four Color Conjecture:**
> **Every map can be colored using four colors.**

first appeared in print in 1878, but didn't become

> **The Four Color Theorem:**
> **Every map can be colored using four colors.**

until 1976, almost a hundred years later.

One interesting feature of discrete mathematics is that it is a mathematical domain where you can reach the frontiers of knowledge quickly and easily. Most of the topics in mathematics that are taught today in schools date back hundreds, even thousands of years, and students and teachers have no idea that mathematics is a living and ever-evolving subject, that there are new problems posed, and new solutions proposed every day. Here, fifty pages into this book, we have an example of a problem that many mathematicians wrestled with for the past 100 years, and that was finally solved only 35 years ago. Later, we will introduce problems for which no solution is known today.

Degree	Number of vertices with that degree	Total edges at those vertices
2	4	8
3	14	42
4	12	48
Totals	**30**	**98**

FIGURE 281

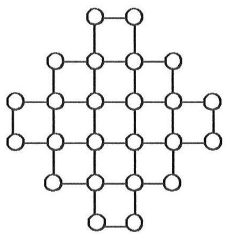

FIGURE 282

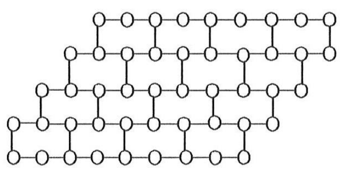

FIGURE 283

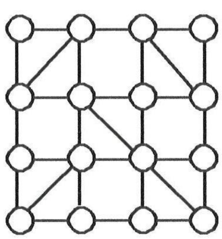

FIGURE 284

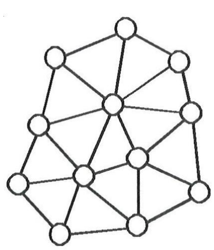

FIGURE 285

In the grid graph $G_{5,6}$ in Figure 280, there are five rows, each with six vertices, for a total of 30 vertices, and there are 49 edges, 25 horizontal and 24 vertical. The 4 corner vertices each has degree 2, the 14 vertices around the border of the graph (excluding the 4 corners) each has degree 3, and the 12 vertices in the center of the graph each has degree 4.

Now of course we could make a table of all 30 vertices and their degrees and add up all 30 numbers, but we can find the sum more quickly if we use some multiplication, as in Figure 281.

This is how the method of "degree counting of edges" works in an arbitrary graph. We create a table whose left-most column has a list of the degrees of vertices in the graph, whose central column has the number of vertices of each degree, and whose right-most column has the product of the degree times the number of vertices of that degree. The sum of the numbers in the right-most column is two times the total number of edges in the graph, since each edge is counted twice.

In the table in Figure 281 we end up with a grand total of 98 edges. Does this equal the number of edges? No, it doesn't. But it is twice the number of edges, which we already know is 49. Why is that? As in the previous section, where we focused on regular graphs, when we count the edges at a vertex, we are counting the *ends* of the edges; since each edge has two ends, each edge is counted twice. So the total number of edges is half the total of the degrees. This is true in any graph.

In any graph,
the sum of the degrees of all the vertices
equals twice the number of edges.

Activity 39: Vertices and Edges

(1) Find the number of vertices and the number of edges in each of the graphs in Figure 282, Figure 283, Figure 284, and Figure 285, first by degree counting the edges using a table like Figure 281, and then, as a check, by counting the edges directly, placing a number on each edge as you count it.

(2) Here's a challenge: Draw a graph that has three vertices of degree 2, three vertices of degree 3, and three vertices of degree 4. Do you think that it can be done?

Go to the Activity Book now, before reading any further, and complete Activity 39.

Chapter 3: Systematic Listing and Counting

Section 3.2: The Handshake Problem ... and Complete Graphs

The Handshake Problem:

If there are a number of people in a room and each person shakes hands with each other person exactly once, then how many handshakes take place?

Let us first consider this problem where the room has 8 people. Then each of the 8 people shakes hands with 7 people, so the total number of handshakes seems to be $8 \cdot 7$, or 56. By similar reasoning, if there were 3 people in the room, then the total number of handshakes would appear to be $3 \cdot 2$, or 6. Take a moment to think about this, and see if you can identify the flaw in this reasoning.

The problem is that with three people – Alfred, Joe, and Louise – there are only 3 handshakes, not 6. Alfred shakes hands with Joe, Joe shakes hands with Louise, and Louise shakes hands with Alfred.

How did we arrive at the answer of 6 in the previous paragraph? We had A shake hands with J and L, J shake hands with A and L, and L shake hands with A and J. That's 6 handshakes altogether.

It can't be that both answers are right!

What happened is that the total of 6 actually counts each handshake twice – the handshake involving Alfred and Joe is counted first as A shakes hands with J and then as J shakes hands with A. Check it out! The handshake involving Alfred and Louise is also counted twice, as is the handshake between Joe and Louise.

So we have to divide the total by 2 to get the actual number of handshakes – for three people, the actual number of handshakes is $(3 \cdot 2)/6$, or 3, and for eight people, the actual number of handshakes is $(8 \cdot 7)/2$, or 28.

All of this should sound very familiar. When we needed to find the number of edges in a graph, we added up the degrees of all the vertices, but we then had to divide by 2, because each edge was counted twice, once at each vertex.

Not only is this familiar – it's exactly the same. For the handshake problem is the same as the problem of finding the number of edges in a complete graph!

Recall that a graph is complete if every two vertices are joined by an edge. The graph in Figure 405, for example, has 8 vertices each of which is linked by an edge to each of the other 7 vertices. This is the complete graph K_8 which appeared in Section 1.10 together with other complete graphs.

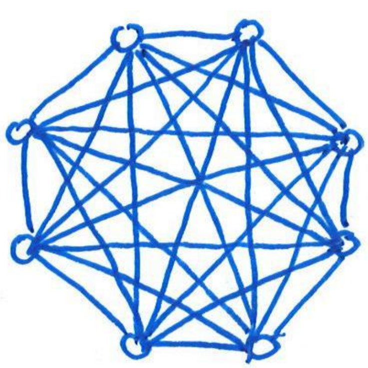

Figure 405

No Impact Whatsoever

The single mathematical statement that would generate the most disbelief in the general population is that **the result of one coin toss (or nine coin tosses) has *no impact whatsoever* on the result of another coin toss!**

Indeed, most people believe that the opposite is true, or, at any rate, act as if they believe the opposite.

This happens each time a person says that they will bet on a certain number because it hasn't come up in the lottery for a month – they will say, "Its time has come!" The fact that the number hasn't come up for a month doesn't make it any more likely that it will come up tomorrow!

This happens each time a person says that they will bet on a certain number in roulette because it hasn't come up all evening. The fact that the number hasn't come up all evening doesn't make it any more likely that it will come up in the next spin of the wheel.

Why is this misconception so prevalent and so persistent. Probably because, on some level, people confuse the two bulleted statements in the other column. They think that because it's so rare for heads to come up ten times in a row that it's equally difficult for that tenth coin to come up heads.

of getting a tenth "heads" is very small, that most likely the next toss will be "tails."

That indeed sounds reasonable, but it is completely incorrect. The probability that the next coin will be heads is exactly 1/2. The fact that the last nine coins all came up heads has no bearing whatsoever on whether the next coin will come up heads or tails!

Make sure that you understand the difference between the following two situations:

- If you have tossed 9 coins and they have all come up heads, what is the probability that the next coin that you toss will also come up heads?
- What is the probability that if you toss 10 coins, they will all turn up heads?

In the first case, 9 coins have already been tossed (and they have all come up heads), but in the second case, no coins have yet been tossed.

A basic assumption about tossing coins is that tossing one coin is independent of tossing another coin and, similarly, that the second toss of a single coin is independent of the first toss of that coin. The coin has no memory of what happened to it the last time it was tossed and is completely unaware of what happened to its fellow coin.

As a consequence, **the result of one coin toss (or nine coin tosses) has *no impact whatsoever* on the result of another coin toss!** (See "No Impact Whatsoever" in the side column.)

It is sometimes easier to solve problems by using the Multiplication Principle of Probabilities than by using choose numbers. Let us go back to our original problems and see how the Multiplication Principle of Probabilities applies to them.

Let us think of selecting two cards from a deck as two separate actions – selecting the first card and then selecting the second card. No matter what card we select first, we are on the right track to getting a pair ... or a flush. So the event E_1 of success on selecting the first card has probability $p(E_1)$ = 1.

If the goal is to get a pair, then success on the second card would be matching the denomination of the first card. That can be done in only 3 ways out of the 51 remaining cards – because when you remove your card from the deck you leave the deck with only 51 cards, of which only 3 match the denomination of your card. Thus the probability $p(E_2)$ is 3/51, or 1/17. Hence $p(E) = p(E_1) \cdot p(E_2) = 1 \cdot 1/17 = 1/17$, the same answer we obtained earlier.

If the goal is to get a flush, then success on the second card would be matching the suit of the first card. That can be done in 12 ways out of the 51 remaining cards – because when you remove your card from the deck you leave the deck

Vocabulary and Notation Review

Addition Principle of Counting – this principle states that if you need to count the number of objects in a certain collection, you can break the collection into smaller groups and count the number of objects in each group. Then the number of objects in the original collection is the sum of the number of objects in all of the groups. (Section 3.1)

Addition Principle of Probabilities – this principle states if an event E can be decomposed into a collection of events $E_1, E_2, E_3, ...$ – that is, each outcome in E is in exactly one of those events – then $p(E) = p(E_1) + p(E_2) + p(E_3) + ...$. (Section 3.5)

choose number – the choose number "n choose m," where m and n are counting numbers, is the number of ways of choosing m objects out of a group of n objects. For example, "8 choose 4" is the number of different pizzas that can be created using 4 of the 8 available toppings, and "18 choose 3" is the number of different ways 3 student representatives can be chosen from a class of 18 students. The **slot method** provides a way of calculating the choose numbers. The choose numbers are also the entries in **Pascal's triangle**. (Sectiion 3.3)

equally likely – see **probability model**. (Section 3.5)

event – see **probability model**. (Section 3.5)

Handshake Problem – the Handshake Problem is the question of how many handshakes take place if each two people in a room shake hands exactly one. If there are n people in the room, the Handshake Problem can be represented as a graph with n vertices where each vertex (that is, each person) is joined by an edge (the "handshake") to each other vertex. This graph is the complete graph K_n, so the Handshake Problem is equivalent to the question of how many edges there are in each complete graph. The answer to both problems is $n(n-1)/2$. (Section 3.2)

Multiplication Principle of Counting – this principle states that if you have a number of tasks T_1, T_2, T_3, ... to perform, and each of these tasks can be performed in t_1, t_2, t_3, ... ways, then the number of ways of performing all of the tasks is the product $t_1 \cdot t_2 \cdot t_3 \cdot ....$ For example, if you have to put together an outfit consisting of a shirt, a skirt, and a pair of shoes, and you have 3 shirts, 4 skirts, and 2 pairs of shoes, then the number of outfits you can wear is $3 \cdot 4 \cdot 2 = 24$. (Section 3.1)

Multiplication Principle of Probabilities – this principles states that if an event E can be thought of as a sequence of independent events E_1, E_2, E_3, ... , then the probability of E is the product of the probabilities $p(E_1) \cdot p(E_2) \cdot p(E_3) \cdot ....$ (Section 3.5)

order doesn't matter – in some counting problems, like the number of ways of selecting two students as class representatives, order doesn't matter, whereas in other counting problems it does – see **order matters**. (Section 3.3)

order matters – in some counting problems, like the number of ways of selecting two students as president and secretary, order matters, whereas in other counting problems it doesn't – see **order doesn't matter**. (Section 3.3)

outcome – see **probability model**. (Section 3.5)

Pascal's Triangle – a triangle formed of numbers, the first 8 rows of which appear below, whose critical properties are that each exterior entry is 1 and that each interior entry is the sum of the two entries above it to the left and above it to the right. The entries in Pascal's triangle are all the **choose numbers** – that is, the m'th entry in the n'th row of Pascal's Triangle is exactly "n choose m." (Section 3.4)

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}$$

FIGURE 462

probability experiment – we can often simulate a theoretical probability model by conducting probability experiments – for example, if three coins are tossed, a **probability model** (see below) will provide a number that represents the probability that exactly two of the coins will land "heads" – for this example, that number is 3/8. However, we can also obtain experimental data by repeatedly tossing three coins – these repetitions are referred to

Addition Principle of Counting

The Addition Principle of Counting, introduced in Section 3.1, says that if you have to count a number of items, you might try to arrange the items in several groups and count the number of items in each group separately. Then the total number of items is the sum of the numbers of items in all of the groups.

When we examine Figure 510 we see that we have arranged all the Hamilton paths in the graph into six groups, depending on the initial vertex of the Hamilton path – i.e., those that start at A, those that start at B, etc. Then we counted the number of Hamilton paths in each group and recorded the results in the table. Finally, we added the results together to get a total of 16 Hamilton paths.

This is a typical application of the Addition Principle of Counting.

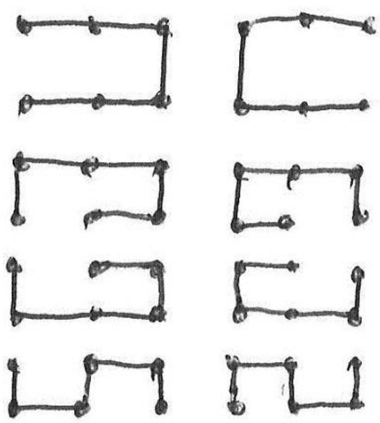

Figure 511

From the table we see that the total number of Hamilton paths in this simple graph is 16. (This is an application of the Addition Principle of Counting – see note in the side column – which is often used without acknowledgement.)

Actually, one could contend that each Hamilton path is counted twice, once in each direction. For example, the Hamilton path A-B-C-F-E-D could be considered to be the same as the Hamilton path D-E-F-C-B-A, since each is the reverse of the other. In that case, since each Hamilton path is counted twice, the actual number of Hamilton paths would be 16/2, or 8. Depending on the circumstances, you could argue that there are 16 or that there are 8 Hamilton paths. In any case, all of them appear in Figure 511.

Activity 64: Hamilton Paths in Graphs

(1) Draw a tree diagram of all Hamilton paths that begin with vertex A in the graph in Figure 512.
(2) Draw a tree diagram of all Hamilton paths that begin with vertex B in the graph in Figure 512.
(3) Draw a tree diagram of all Hamilton paths that begin with vertex C in the graph in Figure 512.
(4) Draw a tree diagram of all Hamilton paths that begin with vertex D in the graph in Figure 512.
(5) Draw a tree diagram of all Hamilton paths that begin with vertex E in the graph in Figure 512.
(6) How many Hamilton paths are there altogether in the graph in Figure 512?

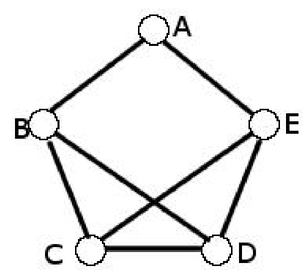

Figure 512

Go to the Activity Book now, before reading any further, and complete Activity 64.

Leonhard Euler

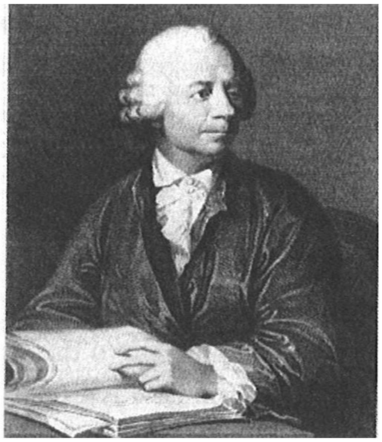

FIGURE 551

Leonhard Euler, gazing at the drawing in Figure 550, was one of the most prominent and productive mathematicians in the 18th century; he is considered to be the originator of the study of graphs that he introduced to solve the Konigsberg Bridge Problem.

He was born in Switzerland, and did most of his work in St. Petersburg and Berlin. "A Short Account of the History of Mathematics" by W. W. Rouse Ball notes that "we may sum up Euler's work by saying that he created a good deal of analysis, and revised almost all the branches of pure mathematics which were then known, filling up the details, adding proofs, and arranging the whole in a consistent form ... Euler wrote an immense number of memoirs on all kinds of mathematical subjects."

We owe to Euler much mathematical notation, including f(x) for a function (1734), e for the base of natural logarithms (1727), i for the square root of -1 (1777), π for pi, and Σ for summation (1755).

We will see another one of his contributions – Euler's Formula – in Section 5.4.

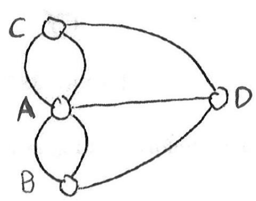

FIGURE 552

The Konigsberg Bridge Problem

Legend has it that in the 17th century, the city of Konigsberg (now Kaliningrad) had seven bridges (see Figure 550) that connected its four land regions (labeled A, B, C, and D in Figure 550) built around the Pregel River. The Pregel has two branches (Alte Pregel and Neue Pregel, that is, old Pregel and new Pregel) and the region between these branches includes an island (region A) called Kniephof. The citizens of the town strolled across the seven bridges on Sunday afternoons, and wanted to know if it was possible to walk across each bridge exactly once and return home. What do you think? Why?

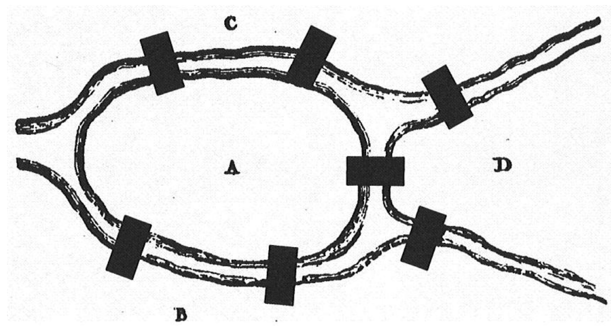

FIGURE 550

Activity 71: The Konigsberg Bridge Problem

Is it possible to take a stroll that will lead you to cross each bridge in Figure 550 exactly once and then return to the region where you began? (Answer this question for each of the four regions, A, B, C, and D, since you could have started your stroll in any region.)

After you have experimented with this problem for 10-15 minutes, write a few sentences presenting and explaining your conclusions.

Go to the Activity Book now, before reading any further, and complete Activity 71.

According to the legend, the Konigsberg citizens couldn't figure out a way of doing this, and sent Leonhard Euler a letter asking if it was possible to take such a stroll. (See the side column for biographical notes and a picture – Figure 551 – of Leonhard Euler.)

Euler answered the question by representing the map in Figure 550 by a graph (see Figure 552) whose vertices are the four regions (the vertex A in Figure 552, for example, represents the island A in Figure 550) and whose edges are the seven bridges, and showing that there was no route that started and ended at the same vertex and that used each edge exactly once. (We'll soon discuss why this is the case.)

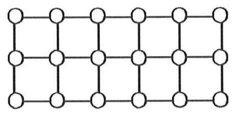

FIGURE 714

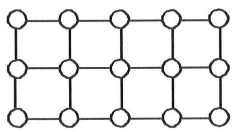

FIGURE 715

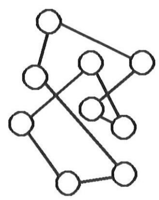

FIGURE 716

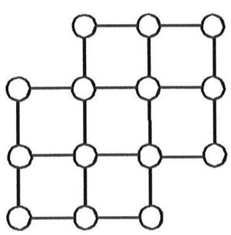

FIGURE 717

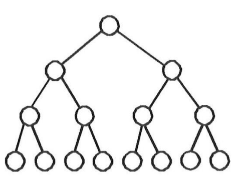

FIGURE 718

- Not every graph has a perfect matching; for example, a graph with an odd number of vertices can't have a perfect matching.
- A maximum matching in a graph is a matching in which as many vertices are paired up as possible; for example, if the graph has an odd number of vertices and you match up all but one vertex, that's a maximum matching.
- A perfect matching is always a maximum matching; since it matches up all vertices, it certainly matches up as many vertices as possible.
- If a graph has a perfect matching, then every maximum matching must also be perfect; if it doesn't match up all vertices, then it can't be a maximum matching since there is a perfect matching that does match up all the vertices.
- If a graph has a maximum matching, it is entirely possible that it doesn't have a perfect matching; it may be that there is simply no way of matching up all the vertices. Even though not every graph has a perfect matching, every graph does have a maximum matching. A maximum matching is, so to speak, a "that's the best I can do" matching.

One way of finding a maximum matching in a graph is to use a Hamilton path in the graph (if you have one). You can walk along the Hamilton path, matching vertices as you go – you match up the first vertex on the path with the second one, the third one with the fourth one, etc.

If the graph has an odd number of vertices, then the last vertex in the path will remain unmatched, but what you will have is a maximum matching. If the graph has an even number of vertices, then all the vertices will be matched and you will have a perfect matching.

Activity 88: Maximum Matching

(1) Find a maximum matching in each of the graphs in Figure 714 through Figure 718. In each case, explain why your matching is a maximum matching and whether it is a perfect matching.

(2) How many different maximum matchings are there in each of the following graphs?

  (a) The chain CH_n

  (b) The cycle C_n

  (c) The wheel W_n

Go to the Activity Book now, before reading any further, and complete Activity 88.

must be that Player 1 has a winning strategy. But no one has ever demonstrated a winning strategy for Player 1!

In 2009, however, according to the Wikipedia article on The Game of Hex, a team of researchers found the winning strategy for Hex on an 8x8 board.

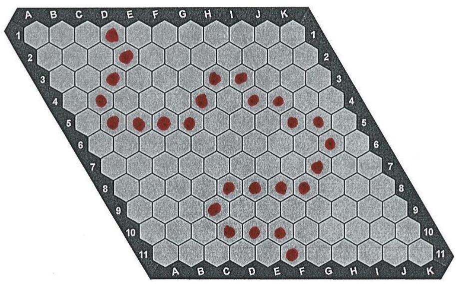

FIGURE 742

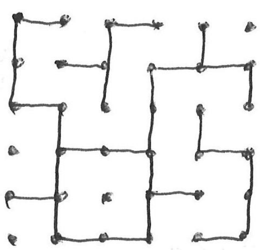

FIGURE 745

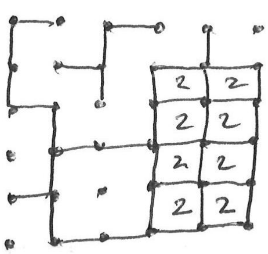

FIGURE 746

FIGURE 747

When the game is over, and all 60 edges of $G_{6,6}$ have been drawn, the player whose initial is in the most boxes wins the game.

For example, in Figure 745, neither player has won any boxes because both players have avoided drawing the third edge of any box. Thus far, 30 edges have been drawn altogether (count them!). Since 30 is even and each player has drawn only one edge at each turn (since there are no boxes), it's now Player 1's turn.

But now whatever Player 1 does allows Player 2 to win one or more boxes. What should Player 1 do?

If Player 1 draws the edge near the center of the bottom row, then Player 2 will draw the fourth edge of that box, then the fourth edge of the bottom right box, then the fourth edge of the box above that one, then the fourth edge of the box to its left, and would end up winning 8 boxes, as in Figure 746. But then Player 2 would have to draw one more edge, so that Player 1 would be able to win some boxes.

But was that the best move for Player 1? Certainly not, because if instead Player 1 drew the edge on the left of the bottom row, Player 2 would have won only one box.

Dots and Boxes is a tricky game! You shouldn't assume that your best move is always to complete all boxes that are available to you. For example, in Figure 747, if your opponent has drawn the middle edge on the bottom row, you would be tempted to complete that box and then the one at the right, However, if you did that, you would then have to draw another edge, and any edge you drew would allow your opponent to win the remaining 7 boxes.

If instead of completing the box, you drew the right edge on the bottom row, you would leave your opponent in a quandary, because no matter what happened you would end up with those 7 boxes.

Thus, it is sometimes better to leave behind a "domino" shape for your opponent, like the two boxes on the bottom center and right of Figure 747, rather than taking those last two boxes. Leaving those two boxes for your opponent may avoid giving your opponent many more boxes.

Activity 92: Dots and Boxes

Assuming that both Player 1 and Player 2 make their best moves in the game of Dots and Boxes, and that they have reached the diagram in Figure 745 with Player 1 to move, which player should win the game, and by what score?

Go to the Activity Book now, before reading any further, and complete Activity 92.

In Dots and Boxes starting with a 6 by 6 array of dots, there are altogether $5 \cdot 5 = 25$ boxes; since 25 is odd, the game can never end in a tie. Therefore, either Player 1 has

Chapter 5: Bipartite Graphs, Matchings, Games, and Planar Graphs

Section 5.4. Planar Graphs

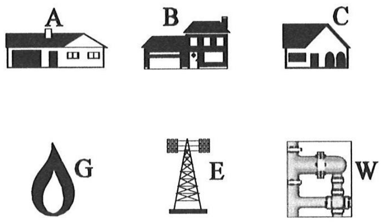

FIGURE 770

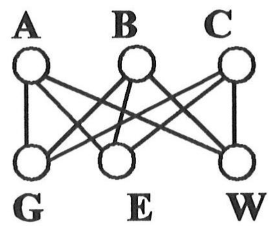

FIGURE 771

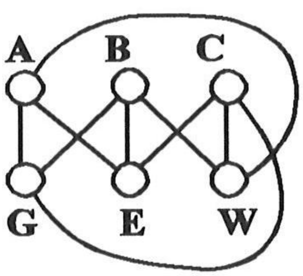

FIGURE 772

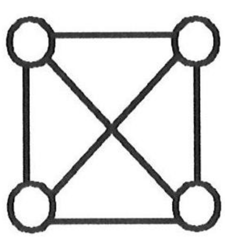

FIGURE 773

A famous puzzle is called the "Utilities Problem". As in Figure 770, there are three houses labeled A, B, and C, and each has to be connected to three utilities – gas (G), electricity (E), and water (W).

The Utilities Problem is: Is it possible to make all the connections in such a way that none of the supply lines cross?

First of all, you should recognize that this question can be modeled using a graph. Each of the six sites can be represented by a vertex, and each of the connections can be represented by an edge, as in Figure 771. The question then becomes: Can you draw this graph in such a way that no two edges cross? Another way of asking this question uses the concept of isomorphism discussed in Chapter 2: Is this graph isomorphic to a graph that has no crossings?

Notice that the graph in Figure 771 is drawn by connecting each house to each utility using a straight line segment. This version of the graph has nine crossings, that is, there are nine places where two edges cross each other. Surely we can do better! Indeed, in Figure 772 we have redrawn the graph of Figure 771 so that there are only 3 crossings; that was accomplished simply by moving the connection from C to G and the connection from A to W. Can you reduce the number of crossings still further?

Sometimes you can redraw a graph with crossings as a graph without any crossings. You may remember that this issue came up in Section 1.10, where we saw that the complete graph K_4 (see Figure 773) can be redrawn as the wheel W_3, a graph where there are no crossings (see Figure 774).

A graph is called a **planar graph** (pronounced "plainer") if it can be drawn without any crossings – that is, if it is isomorphic to a graph that has no crossings.

Imagine that the edges of a graph are strings. One way of ensuring that the complete graph K_4 is drawn without any crossings is to lift up one of the crossing edges so that it is no longer in the same plane as the other edges. More generally, any graph can be drawn without crossings if three dimensions are used. Thus, for example, those who install the utility supply lines ensure that no two supply lines actually cross by placing one of the supply lines above the other, that is, in a different plane.

But when you can ensure that no two edges cross and still stay in the plane, that's something special – that's when a graph is really "planar". The property of "being planar" is referred to as **planarity**.

A Review of Spanning Trees

Recall that a spanning tree in a graph is a set of edges of the graph that form a tree that includes every vertex of the graph. Recall also that a tree is a connected graph with no cycles. Because a tree is connected, a spanning tree provides a route from any vertex to any other vertex.

On the other hand, because a tree has no cycles, we are able to conclude from the discussion in Section 4.5 that a spanning tree provides *only* one route from any vertex to any other vertex.

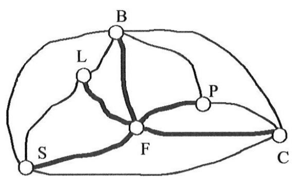

FIGURE 838

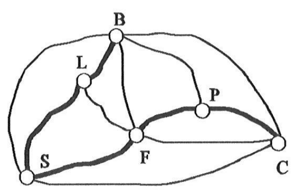

FIGURE 839

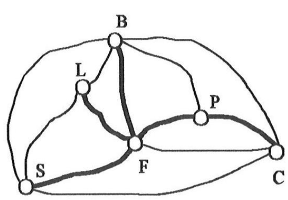

FIGURE 840

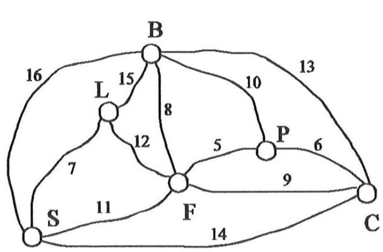

FIGURE 841

in Figure 838, Figure 839, and Figure 840, defined in each case by the edges that are bold. Which one gives the best solution?

We could of course argue that the best solution is the one that links every site directly to the firehouse (as in Figure 838), since then the fire trucks could get to the other five sites most quickly. Or we might argue that a linear arrangement (as in Figure 839) is fairest, or most efficient, since you can travel to all the sites along a single road.

But usually the decision is based on quantitative information – the "bottom line" – what is the cost? The five roads selected for paving will most likely be the five roads which do the job (that is, connect all the sites) and whose total cost is as inexpensive as possible.

We therefore imagine that the Muddy City Council first obtains estimates for paving each of the twelve roads. These estimates are recorded as numbers on the edges of the graph in Figure 841; all are multiples of $100,000. Note that the cost for paving each road depends not only on its length, but also on other factors, such as hills, curves, and drainage; that's why a road that appears to be longer might involve a lower cost than a road that appears to be shorter.

Activity 100: The Muddy City Problem Revisited

The graph in Figure 841 represents a map of Muddy City that shows its six critical sites and all the unpaved roads that connect them. Each unpaved road is labeled with a number that represents how much (in multiples of $100,000) it will cost to pave it.

Which five roads should be paved in order to minimize the total cost but, at the same time, ensure that you can travel between each two sites on paved roads?

Go to the Activity Book now, before reading any further, and complete Activity 100.

A spanning tree that has minimal weight is referred to as a **minimum weight spanning tree**. Finding a minimum weight spanning tree in a graph is referred to as **the minimum weight spanning tree problem** for that graph. The Muddy City Problem in Activity 100 is an example of a minimum weight spanning tree problem, as is the problem in Activity 101.

A less formal way of describing the minimum weight spanning tree problem for a graph is to ask, "What's the cheapest network?" or "What's the cheapest way of linking these sites into a network?" In this context, the term "network" is treated as a synonym for "connected subgraph".

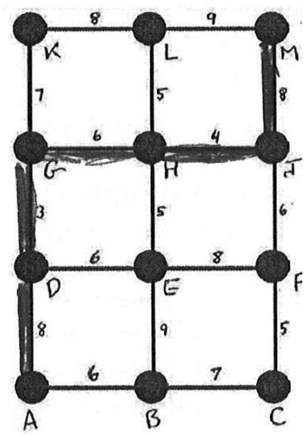

FIGURE 933

Edsger Dijkstra

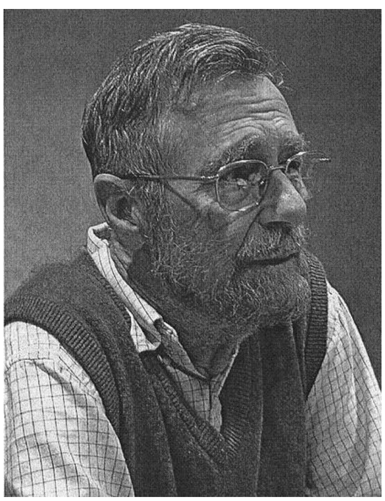

FIGURE 934

Edsger Wybe Dijkstra (1930-2002) was a Dutch computer scientist. He was a professor at the Eindhoven University of Technology in the Netherlands, a Research Fellow for the Burroughs Corporation, and, between 1984 and 2002, a professor at The University of Texas at Austin. An obituary by Krzysztof R. Apt notes that:
"Through his fundamental contributions Dijkstra shaped and influenced the field of computer science like no other scientist. Many of his papers, often just a few pages long, are the course of whole new research areas. Even more, several concepts that are now completely standard in computer science were first identified by Dijkstra and bear names

– in this situation, we had to consider all of the possibilities and not focus only on what seems best at the moment.

Activity 112: The Shortest Route

(1) Use Dijkstra's Algorithm to find the route from A to B in the graph in Figure 935 that has the shortest total distance.

Note that, as discussed above, to solve this problem you will need to actually find the shortest route from A to every vertex in the graph; if your solution doesn't consider every vertex, then it may not give the shortest route from A to B – it may just happen that the shortest route passes through one of the vertices that you didn't consider.

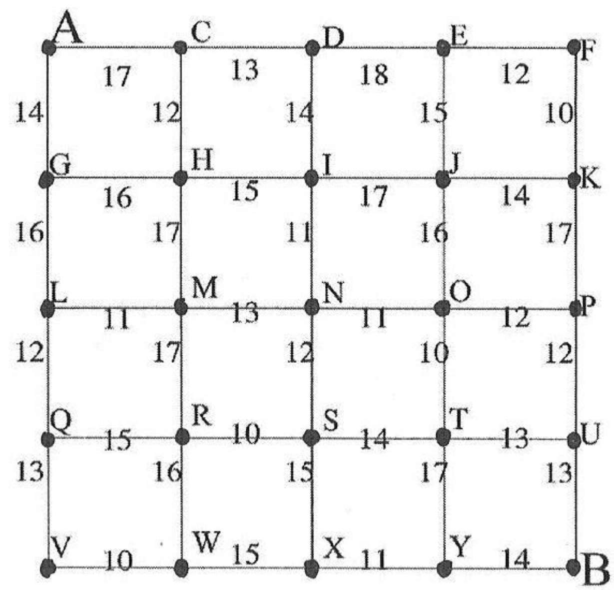

FIGURE 935

(2) Use Dijkstra's Algorithm to find the route from Poughkeepsie to Hanover in the graph in Figure 923 that has the shortest total distance.

Go to the Activity Book now, before reading any further, and complete Activity 112.

Chapter 6 – Summary

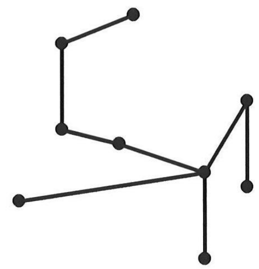

FIGURE 952

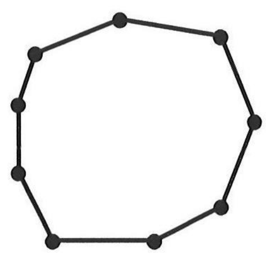

FIGURE 953

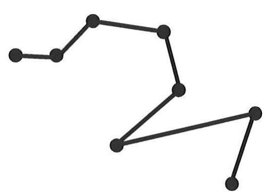

FIGURE 954

In Sections 6.1, 6.2, and 6.3, we considered three types of problems related to weighted graphs. We review here the similarities and differences of these problems and their solutions.

As we noted at the beginning of Section 6.3:

- In Section 6.1, we considered the problem of finding the **shortest network** that linked a number of sites; we were looking for a **minimum weight spanning tree**, as exemplified by the graph in Figure 952.
- In Section 6.2, we considered the problem of finding the **shortest circuit for a traveling salesperson**; we were looking for a **minimum weight circuit** that visited each site and returned to the starting point, as exemplified by the graph in Figure 953.
- In Section 6.3, we considered the problem of finding the **shortest route from home to school**; we were looking for a **minimum weight path** that started at one site and that ended at a second site, as exemplified by the graph in Figure 954.

What is common to these three types of problems?

- Each problem deals with weighted graphs.
- In each problem, we are looking for a particular type of subgraph of the given graph.
- Each problem is an optimization problem; that is, in each case there are a number of possible subgraphs and we want to find one that is optimal, in that the total weight of the edges in the subgraph is as small as possible.

How are these problems different?

- The type of subgraph that we seek is different.
  - In the shortest network problem, we are looking for a spanning tree (see Figure 952).
  - In the Traveling Salesperson problem, we are looking for a circuit (see Figure 953).
  - In the shortest route problem, we are looking for a path (see Figure 954).
- The result of our search is different.
  - In the shortest network problem, a greedy algorithm always gives us an optimal solution. Indeed, there are several different greedy algorithms that work, for example, Kruskal's Algorithm in which the cheapest unused edge is added so long as it doesn't result in the creation of a circuit.
  - In the shortest route problem, a greedy algorithm cannot work, for the shortest route could conceivably include any vertex, so we have to consider all of the possibilities. However, there

The Huntington-Hill Method

The main feature of Daniel Webster's method of apportionment is that we round normally, that is to say, if the decimal part of the number is .5 or greater, we round up to the next highest integer, whereas if the decimal part of the number is less than .5, we round down to the next lower integer.

For example, if the number is 7.5 we round up to 8, whereas we round 7.49 down to 7. The borderline between rounding up and rounding down is always exactly in the middle – whether it's 7.5 or 83.5 or 521.5. The borderline is always the average between the next higher number and next lower number – thus, the average of 7 and 8 is 7.5, the average of 83 and 84 is 83.5, and the average of 521 and 522 is 521.5.

In the Huntington-Hill Method, the borderline is not the numerical average of the two numbers, but the geometric average – which is obtained by multiplying the two numbers and taking their square root.

For example, the geometric average of 7 and 8 is $\sqrt{(7 \cdot 8)} = 7.4833$, the geometric average of 83 and 84 is $\sqrt{(83 \cdot 84)} = 83.4985$, and the geometric average of 521 and 522 is $\sqrt{(521 \cdot 522)} = 521.4998$.

As you can see, for larger numbers, the geometric average is almost the same as the numerical average, whereas for small numbers, the geometric average is several percentage points less than the numerical average.

Thus, if we are using Webster's method of rounding normally and using the geometric average rather than the numerical average as the borderline, then we are slightly favoring the smaller states.

The Huntington-Hill method, known as "the method of equal proportions," is due to mathematician Edward Huntington and statistician Joseph Hill.

Credits

This section on "Apportionment" is based on a presentation made by Ronald (Chuck) Tiberio, a mathematics teacher at Wellesley (MA) High School; Chuck was a participant in the 1992 Leadership Program in Discrete Mathematics at Rutgers University.

The data of the main example come from a presentation he attended many years ago in the Boston area.

In 1931, the method of apportionment was changed to the Huntington-Hill method (see note in side column), which is a slight modification to Webster's method. The Huntington-Hill method provides a small benefit to the small states.

Is there a perfect system for apportionment?

The answer is no. In 1982, two mathematicians, Michel Balinski and Peyton Young, proved that any method of apportionment involving three or more states that satisfies the "quota rule" – that is, each state gets either the maximum or the minimum – will result in paradoxes.

Activity 121: Apportionment

The population of the six towns in Monroe County is given in Figure 996.

Town	Population	Rightful # of Seats
Lincoln	4,455	8.910
Johnson	3,294	
Grant	6,612	
Harrison	2,424	
Cleveland	3,976	
McKinley	4,249	
Total	25,000	50

FIGURE 996

(1) The County Coordinating Committee (CCC) has 50 members. Determine the rightful number of seats each town should have. For example, since Lincoln's population is 4,455, and the number of people per seat should be $25,000/50 = 500$, its rightful number of seats is the number of times 500 divides into 4,455, which is $4,455/500 = 8.910$.

(2) Use each of the five methods to determine how the seats on the CCC may be fairly apportioned among the six towns. (Note that you will need to divide by numbers other than 500 by assuming that the number of people per seat is more or less than 500.)
   (a) Alexander Hamilton's method
   (b) Thomas Jefferson's method
   (c) Daniel Webster's method
   (d) John Quincy Adams' method
   (e) Huntington-Hill method

Go to the Activity Book now, before reading any further, and complete Activity 120.