Discrete Mathematics in 21st Century Education:

An Opportunity to Retreat From the Rush to Calculus

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Abstract

One way of characterizing K-12 mathematics education in the second half of the 20th century is that it has been a rush to calculus. Perhaps the first half of the 21st century will be seen as a time of retreat from that rush, when, at the high school level, mathematics is enriched by new topics like probability, statistics, and discrete mathematics; traditional topics that lead to calculus are treated more deliberately and more thoroughly; and these perspectives are reflected throughout the curriculum. This paper anticipates that discrete mathematics topics will play a larger role in the K-12 curriculum, as more educators learn these topics and recognize their value in achieving the goal of improving students' understanding of mathematics.

The Rush to Calculus

The phrase "rush to calculus" captures one significant aspect of the changes in mathematics education at the high school level over the past 50 years.

At the high school that I attended and graduated from in 1957 – Benjamin Franklin High School in Rochester, New York – I took trigonometry and solid geometry in my senior year and, at the end of the year, took the New York State Regents Examinations in those subjects. When I enrolled at Columbia College that fall, the honors class that I attended began with analytic geometry; my classmates included graduates of Bronx Science and other selective schools, so my mathematical background was not unusual. By today's standards we would be at least three semesters behind[1]; indeed, 73% of the 61 students in my first-semester calculus class in Fall 2002 at Rutgers University had taken at least one year of calculus in high school.

What changed? Certainly a major contributor was the introduction and popularization of the Advanced Placement Examination in Calculus. "Advanced Placement" is an alluring title and an alluring idea. Any parent would love to hear that his or her child is "advanced" and to have that confirmed by the child's acceptance into a course that is nationally recognized as "advanced". Moreover, there is the promise that the child would start college ahead of his or her peers, take even more advanced courses earlier, and maybe even complete college work ahead of schedule.

Here is how the Advanced Placement Program is promoted on its website: "AP can change your life. Through college-level AP courses, you enter a universe of knowledge that might otherwise remain unexplored in high school; through AP Exams, you have the opportunity to earn credit or advanced standing at most of the nation's colleges and

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[1] Starting college "three semesters behind" did not lead to mathematical disaster for me. Indeed, I earned a Ph.D. in mathematics and continued mathematical research for fifteen years before focusing my attention on improving K-12 mathematics education.

universities.” Moreover, the website indicates that by taking AP courses and examinations, students will “gain the edge in college preparation, stand out in the college admissions process, and broaden your intellectual horizons.”

Not only will you “gain the edge in college preparation”, but you will also “stand out in the college admissions process”. I don’t know whether this is true in general, but admissions officers at seven major state universities called in May 2005 by Rutgers graduate student Allison McCulloch all indicated that AP Calculus would be considered more favorably that any other high school mathematics course. Interestingly enough, they all indicated that scores on the AP Calculus examination had no weight in their decisions, presumably because test scores would not arrive until well past the time for admissions decisions. Thus simply taking the AP Calculus course is considered a positive factor by college admissions officers.

This practice is called into question by a recent “major study by researchers at the University of California at Berkeley” that shows that “college-level courses offered in high school, such as Advanced Placement (AP) or International Baccalaureate (IB), do not appear to improve academic performance in college, unless students take the tests at the end of each course.”[2] The illusion has been established, however, that simply taking AP Calculus leads to success.

A second major contributor to the rush to calculus is that parents all around the country have somehow come to the conclusion that taking calculus, even if it’s not AP calculus, will make a decisive difference in getting their children admitted to college.

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[2] As reported in the Washington Post, December 23, 2004.

That conclusion was not supported by the calls to the admissions officers; indeed, the general sense seems to be that taking a non-honors course in calculus would usually not make much of a difference, although taking mathematics for four years was clearly preferred. How important calculus is in the admissions process is thus not very clear, and is a question that bears investigation.

Another question that bears investigation is what percentage of students who take calculus actually obtain advanced standing. Nearly 20 years ago, long before college admissions were handled electronically, I physically went through the folders of hundreds of Rutgers College first-year students to see what they had taken in high school. I do not know what happened to the raw data that I collected, but what I remember clearly is that:

- About half of the students who took calculus in high school took an AP course,
- About half of those who took the AP Calculus course took the exam,
- About half of those who took the AP Calculus exam received grades of 4 or 5 (sufficient to get college credit), and
- About half of those who received grades of 4 or 5 actually continued on an accelerated path.

That means that about 1 out of 16 students who were accelerated into a calculus course in high school actually took advantage of that acceleration by earning and making use of advanced placement in mathematics. So for the students who took calculus in high school, the main outcome of their acceleration was that when they got to college they could slow down and repeat calculus. That does not make much sense.

I am currently engaged in a more careful look at more recent data. Of the 2120 students entering Rutgers College in the fall of 2003, exactly 332 (or 15.7%) took the AP Calculus exam (or, more precisely, Rutgers received copies of their scores). Of these students, 161 (just less than half) received college credits; 112 received 4 credits and 49 received 8 credits. This data is consistent with the third bullet in the previous paragraph. How many of those 161 students continued on their accelerated track? For those who received 4 credits, an appropriate definition seems to be that they successfully completed second and third semester calculus in their first year at Rutgers; 39 did and 72 did not. For those who received 8 credits, an appropriate definition seems to be that they successfully completed two of the three second year mathematics courses (multivariate calculus, linear algebra, differential equations) in their first year at Rutgers; 26 did and 23 did not. Altogether, of the 161 students who were able to start college on an accelerated path, only 65 continued that acceleration. These findings are consistent with the final bullet above.

The second part of the study involves examining the high school transcripts of 335 students randomly chosen from the cohort of 2120 students described above. Of these 335 students, 105 students (29.6%) took an AP calculus course in high school and an additional 82 students (24.5%) took a calculus course that was not an AP calculus course. This is consistent with the first bullet above; although more schools are offering and more students are taking AP calculus courses, more students are also taking non-AP calculus courses; altogether over half of our incoming students have taken a full-year calculus course, which was certainly not the case 20 years ago. Of the 105 students whose transcripts indicated that they took an AP calculus course, only 51 appeared on the list of students who took the AP calculus exam; the other 54 did not. This is consistent with the second bullet above.

Altogether, we can say that 105/187 of the students who take calculus in high school actually took an AP calculus course, that 51/105 of those who took an AP calculus course took the AP exam, that 161/332 of those who took the exam received credits for calculus, and 65/161 of those who received credits continued in their accelerated path. Combining these fractions, we conclude that fewer than 1 out of 18 (5.3%) of those who took calculus in high school continued in their accelerated path. These results corroborate the earlier informal findings, but these results still do not make sense.

Now one might argue that high school students who take calculus do in fact "gain an edge in college preparation" even if they take calculus over again in college. The reasoning is that the second time you see the material you are able to consolidate and move beyond what you learned the first time. However, the problems first-year students have with calculus are not generally with the concepts of calculus but with the manipulations of algebra. For example, in evaluating the expression for the derivative of the function $f(x) = 1/x$, the expression $[1/(x+h) - 1/x]$ arises. Some students will not recognize that the appropriate step is to combine the two fractions into a single fraction, others will not know how to combine two fractions, and many who are aware of the procedure will implement it incorrectly. Students often make errors that reflect misunderstandings of the fundamentals of algebra (such as inappropriately canceling terms in fractions). Taking calculus a second time will not correct these misconceptions;

a more demanding high school algebra course might be a better strategy. It is no wonder that many college teachers would prefer their students to come to college with stronger algebraic skills and understandings than with a background in calculus.

In order to take calculus as seniors, students take an accelerated math program that starts with Algebra I in grade 8, and continues with Geometry in grade 9, Algebra II in grade 10, Precalculus in grade 11, and Calculus in grade 12. We have already seen that only a small percentage of the students who take calculus benefit from this acceleration and that most indeed come to college with insufficient understanding of the topics of high school mathematics. What about those who do not eventually take calculus in the 12th grade? It is true that many students seem to be ready to take Algebra I in the eighth grade; however, many of them will not be ready for Algebra II in the tenth grade. At one school where we interviewed students a number of years ago, we found that students who thought that they were mathematically competent hit a wall when they took Algebra II in the tenth grade. The consequence of their taking Algebra I earlier than necessary was that they ended up dropping out of math.

Thus the rush to calculus has a negative effect on the students who do not get to calculus as well as on those who do. And by making calculus the norm for college-bound students, the rush to calculus ends up distorting the entire high school curriculum, so that students chase after calculus rather than learning algebra and geometry well and studying mathematics topics that might be more valuable to them, like probability, statistics, and discrete mathematics. The rush to calculus, and the resulting rush *through* algebra, geometry, and trigonometry, produces students who have a myopic view of those subjects, that is, they may have an ability to carry out narrowly prescribed operations, but do not have the big picture.

This phenomenon is likely to be exacerbated by the current fad, a rush to algebra. More and more students are being encouraged to take a formal algebra course in grade 8, despite the fact that many educators believe that not all students are cognitively ready for a formal algebra course in grade 8. The initiative that all children should take Algebra I in grade 8 has a noble goal, that of equity. It is apparently based on the idea that calculus in high school is a positive goal and, from the perspective of equity, should therefore be a universal goal. Unfortunately, because of the state of elementary and middle school mathematics education in the United States, most students are not prepared to take a formal algebra course in grade 8 and most 8th grade mathematics teachers do not have the mathematical background to provide students with a strong grounding in algebra. For all these reasons, the attempts by states and districts to ensure that all students take a traditional Algebra I course in the 8th grade are, in my opinion, misguided[3].

We pay a big price for rushing to calculus and rushing through algebra, with very little payoff. We accelerate all students so that a few can earn college credits for calculus, and then only a small percentage of those who are in a position to benefit from those credits actually do so. We develop curricula that focus on calculus, depriving

[3] Writing on "Pushing Algebra Down" in the March 2005 News Bulletin of the National Council of Teachers of Mathematics, President Cathy Seely notes that "students should take algebra early only if they are highly motivated to do so, only if they intend to study mathematics through the calculus or statistics level, only if the system is structured to accommodate this advanced study, and only if the student also studies proportionality and the most important components of a good middle school program. Far more important than *when* they study algebra is *what* they study and *whether* they are taught in a way that helps them learn it and use it for the long term."

students of the opportunity to learn other mathematical topics and to gain mastery of algebra and geometry.

What are the Alternatives?

As noted in the abstract, at the high school level mathematics can be enriched by new topics like probability, statistics, and discrete mathematics, and traditional topics that lead to calculus can and should be treated more deliberately and more thoroughly. At elementary and middle school levels, students should be learning other topics as well as the traditional ones. The rush to and through algebra should be decelerated.

In this chapter I will concentrate on the discrete mathematics option since that has been one of my main focuses for the last fifteen years.[4]

Some may ask, using language that emerged from the discussions of the Third International Mathematics and Science Study (TIMSS) Report, whether our mathematics curriculum is already "a mile wide and an inch deep"? That is, are we not already including too many topics in our mathematics curriculum and not going into them at sufficient depth? Certainly the lack of depth and thoroughness is a problem, as noted earlier. However, the argument that we should reduce the number of topics is not convincing. The set of topics covered in the TIMMS assessment were those common to the curricula of all of the participating countries. So topics such as probability, statistics, and discrete mathematics were excluded from consideration. It would not be unexpected

[4] The efforts over the past 25 years to introduce discrete mathematics into schools in the United States are described in DeBellis and Rosenstein (2004).

for the performance of American students on the common topics to be slightly lower than in other countries, where 100% of instructional time was devoted to those common topics. If our goal were success on the TIMSS assessment, then we could argue for reducing the number of topics. But why should we set our sights so low, when we could expect students to achieve a broader and deeper understanding of more topics than what is expected in other countries?

What is Discrete Mathematics?

Unfortunately, there is no simple answer to this question. Unlike algebra, geometry, probability or calculus, discrete mathematics is not a well-defined area of mathematics. One could contrast "discrete" with "continuous": Whereas the functions in calculus (and, more generally, real analysis) typically have domain consisting of all real numbers (the "continuum"), the domains of functions in discrete mathematics are typically the natural numbers; their graphs are discrete sets of points rather than continuous curves. But dividing all of mathematics into "discrete" and "continuous" places too many topics in the "discrete" basket. There is no general agreement on which topics should actually be included in discrete mathematics; indeed, an article by Stephen Maurer in *Discrete Mathematics in the Schools*[5] is called "The Many Definitions of Discrete Mathematics."[6]

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[5] Two important collections of articles about discrete mathematics in K-12 education are *Discrete Mathematics in the Schools* (1996) edited by Joseph G. Rosenstein. Deborah Franzblau, and Fred Roberts, and *Discrete Mathematics Across the Curriculum K-12*, the 1991 NCTM Yearbook, edited by Margaret Kenney and Christopher Hirsch.

[6] Maurer explores and rejects four approaches that attempt to define discrete mathematics by "specifying the properties", including, for example, "discrete mathematics is the mathematics of discrete sets" and "discrete mathematics is any mathematics that doesn't involve limits." He also provides lists of topics for

Instead of trying to *define* "discrete mathematics" we may agree on some areas that it includes. Indeed, the *Principles and Standards for School Mathematics* (PSSM) of the National Council of Teachers of Mathematics (NCTM) recognizes three particular areas – vertex-edge graphs (often referred to as "networks"), combinatorics (that is, systematic counting), and iteration and recursion (modeling change discretely) – as important at all K-12 grade levels, and adds matrices as an important area at the high school level. However, several important areas are omitted from PSSM, like the constellation of topics that are related to fairness and social choice (e.g., apportionment, elections, fair division) and the topics that are related to information (e.g., codes, sorting).

In any case, discrete mathematics is a wide-ranging collection of mathematical topics including coloring maps, finding shortest routes, scheduling tournaments, constructing fractals, conducting elections, sorting, and counting systematically. Here is a list of simply-stated problems that fall under the rubric of discrete mathematics, following the order of the five topics in the preceding paragraph:

Vertex-edge graphs

- Which way of connecting a number of sites into a network involves the least cable?
- What's the best way for a robot to pick up items stored in an automated warehouse, or for a courier to collect deposits at all ATM machines in the assigned region?

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five different textbooks that focus on discrete mathematics. Finally, he discusses the possible ways of grouping discrete mathematics topics by their emphases and by their goals.

- What is the smallest number of colors needed to color the 48 states in the continental United States if states that share a border must be colored with different colors (so that all borders can be clearly distinguished)?

Combinatorics

- How many different pizzas can you have if each pizza must have at most three of the eight available toppings?
- How many tickets do you have to buy to make sure that you have a winning ticket in the contest that involves correctly selecting six numbers from 1 to 36?

Iteration and recursion

- What should be the daily dose of medication if, to function effectively, the medication must be at a specified concentration and if a given percentage of the medication in the body is eliminated each day?
- If the population of deer increases by 10% each year, how long will it take the population of deer to double?

Social choice

- What is the best system for reapportioning the 465 seats in the United States Congress among the states after each census? What system is actually used?
- What is a good strategy for dividing up a pie among three people so that each is satisfied with the portion he or she receives?

Information

- What is the quickest way of alphabetizing a list of 1000 names – on index cards, or in a database?
- How are transmission errors detected and corrected when coded versions of pictures are sent from space?

If discrete mathematics is so important, why didn't we learn about it when we went to school? One simple answer is that, when we went to school, many of these problems were computationally too difficult to solve. Consider for example the problems in the second bullet above, variations of what is traditionally called the Traveling Salesman Problem. If the courier has to collect deposits from ten ATMs, she can do that in any one of 10! possible orders since she can visit any one of ten sites first, then any one of the remaining nine sites, then any one of eight sites, etc. To determine which route would be shortest (or would take the least time) would require an examination of over three million possible routes. It is only in the age of computers that such a task might be possible. Once computers became available, the problems of discrete mathematics achieved a greater degree of importance, both practically and theoretically.

Why Should Discrete Mathematics be Part of the K-12 Curriculum?

One reason that discrete mathematics should be part of the K-12 curriculum is that it is widely used. It is a rapidly growing and increasingly used area of mathematics with many practical and relevant applications, particularly in the technological and information sciences. Vertex-edge graphs are used, for example, to model the networks that underlie telecommunications systems and the flow charts that present a complex project in terms of simpler components. Combinatorics is fundamental to probability and to modern applications of codes. Iteration and recursion are the mathematical ideas that are basic to the study of discrete change in economics and biology. Discrete mathematics is grounded in real-world problems (like those in the list above), and, as noted in NCTM's *Principles and Standards for School Mathematics*, "As an active branch of contemporary mathematics that is widely used in business and industry, discrete mathematics should be an integral part of the school mathematics curriculum" (p. 31).

That discrete mathematics is widely used obviously has implications for those students who anticipate using mathematics in their future endeavors, since it is likely that the applications that they will encounter will involve the use of discrete mathematics. However, the wide use of discrete mathematics also has implications for those students who may not use mathematics and even for those students who do not anticipate going to college. As informed citizens and consumers, they should be aware of problems like those listed above and how mathematics can help solve them.

A related reason for including discrete mathematics in the high school curriculum is that it provides ready answers to the question that high school students repeatedly ask: "What is all this mathematics good for?" Many of the applications of discrete mathematics are accessible to all high school students. Students can see the applicability of algebra and geometry, but those on the calculus track have to complete two additional years of preparation – algebra II and precalculus – and a semester of calculus before they see their first applications of calculus to optimization problems (maximum and minimum problems). Discrete optimization problems can be introduced much earlier. For example, once you introduce vertex-edge graphs, you can add weights (representing distance or time or cost) to the edges and ask, “What is the shortest path from A to B?”

This brings us to a third reason for including discrete mathematics at the high school level – discrete mathematics problems can be introduced without much preparation. All students can understand the question posed above, and if provided a “map” like that in Figure 1, where the numbers represent distances, can find the shortest path from A to B by trial-and-error or by a systematic analysis of all possible routes.

Insert Figure Rosenstein.1 about here

Moreover, they can understand and apply (although probably not discover) an algorithm that will indeed provide the shortest path in any such map. What mathematical content knowledge is prerequisite for this activity? Only addition of whole numbers (Of course, if the distances were given as fractions or decimals, then the ability to add such numbers would be needed; on the other hand, shortest path problems provide a painless way of reinforcing addition skills for such numbers). That discrete mathematics problems can be introduced with little preparation has two important consequences.

First, it can be used to provide students with a different view of mathematics. Through many years of focus on the mechanics of mathematics, students come to equate mathematics with formal arithmetic and algebra. For them, mathematics is not a domain where big ideas are discussed, or where creativity is exercised. Although students may see applications of mathematics in their science courses, those applications are again likely to involve routine calculations (like balancing equations in chemistry). They are not exposed to mathematical modeling in a variety of arenas or to the variety of tools that mathematicians use to analyze everyday situations. Discrete mathematics is an arena where mathematical modeling is easily understood. If you want to make sure that your yearbook is completed before graduation, you can use vertex-edge graphs to analyze the collection of tasks that need to be done and then use critical path analysis to determine the date by which each task must be completed.

Second, for those students who have not been successful in mathematics, discrete mathematics offers a "new start" – it provides mathematical topics that do not have algebraic skills as a prerequisite, but that rely mainly on reasoning and problem solving. For these students, discrete mathematics offers an arena where they can be successful in mathematics without realizing that they are doing mathematics (since students often equate mathematics with arithmetic and algebra), where they can overcome the stigma of repeated failure, and with newly generated self-confidence can find perhaps that their stumbling blocks to learning traditional topics were not insuperable. Enrichment through discrete mathematics has perhaps more potential than repeated remediation for these students.[7]

[7] For further discussion of the "new start" that discrete mathematics provides for students, see the articles by Biehl, Picker, and Rosenstein in the volume *Discrete Mathematics in the Schools*.

Another quality shared by many topics of discrete mathematics is that they are engaging – the problems are more visual than computational, more geometric than algebraic, and they often have a puzzle-like quality – and students easily get engaged in them. Thus discrete mathematics is a very attractive alternative for those students who are moderately successful at traditional mathematics, but are not fans of mathematics. They struggled successfully with fractions and algebra, but did not find them fun. Discrete mathematics provides an opportunity for them to enjoy mathematics again[8]. And students find discrete mathematics challenging because, while many problems in discrete mathematics are easily stated, they are often not easy to solve.

Indeed these qualities of discrete mathematics make it particularly appealing to those students who are really adept at mathematics. From the outset, challenging problems can be presented, including problems whose answer is unknown. Students can quickly learn about the P vs. NP Problem, whose solution will earn $1,000,000 for its solver from The Clay Mathematics Institute of Cambridge, Massachusetts (CMI). This is one of seven "Millenium Problems" named by CMI in 2000 to celebrate mathematics in the new millennium. Given what has been said about discrete mathematics in this chapter, it is not surprising that this problem is the only one of the seven Millenium Problems that is accessible to high school students[9].

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[8] The opportunity that discrete mathematics offers to make mathematics enjoyable for students can only be realized when they are actively engaged in the learning of the mathematics. This point is discussed further in the penultimate section of this article.

[9] For a discussion of the P vs. NP problem, and the other "Millenium Problems", visit the website http://www.claymath.org/millennium/.

These qualities of discrete mathematics – that it has many applications, that it provides ready answers to the question, "What is mathematics good for?", that many of its topics have few prerequisites, that *all* students find it engaging, and that it presents challenges to the strongest students – provide strong justification for why discrete mathematics should be included in the high school curriculum.

Why Should Discrete Mathematics be Included in the K-8 Curriculum?

Although the qualities of discrete mathematics discussed above apply also to some extent at the K-8 grade levels, there are other reasons as well for the inclusion of discrete mathematics in the K-8 curriculum. Let us reflect for a moment on the statement in NCTM's *Principles and Standards for School Mathematics* cited above. The statement recognizes the value of the *mathematics* in discrete mathematics. Unfortunately, it does not recognize the *educational value* of discrete mathematics. And that is the critical aspect of discrete mathematics that makes it valuable at the earlier grade levels.

Children need to be able to understand and use a variety of concepts and techniques from different areas of mathematics, and to be able to solve problems that draw on their toolbox of mathematical and problem-solving strategies. Learning the basics is no longer enough. Today's children live in a technological age where they will need to think critically, solve problems, and make decisions using mathematical reasoning and strategies. The mathematics standards, both state and national, expect students to go beyond the basics – in terms of additional content and substantially increased problem-solving and reasoning, and in terms of applying school math to daily situations.

A major obstacle to students' going beyond the basics is that their teachers often never went beyond the basics. Many K-8 teachers have taken few courses in mathematics, and have never themselves been engaged in a single real problem-solving mathematical activity or been asked to explain or justify their reasoning mathematically. They use worksheets to provide lots of practice to their students, thinking that they are engaging their students in problem solving, not realizing that a problem is different from an exercise; what makes a problem “a problem” is that you do not recognize immediately how to solve it.

Moreover, since they use only direct instruction (the counterpart of lecturing at the college level), they often do not engage their students in the learning of mathematics; students are expected to recite mathematics (such as memorized facts), manipulate symbols, or apply formulas, but are rarely expected to be actively engaged in figuring out the answers to more difficult questions or in verifying their claims. Those students whose learning style thrives on direct instruction do well; not so, however, for the substantial numbers of students who need to build their own understanding of a concept or technique. So teachers who have never gone beyond the basics cannot reach *all* students, as recommended in the standards.

Discrete mathematics offers an entree into mathematics for teachers who have not been successful at mathematics. It offers them an opportunity to see what mathematics is all about, an opportunity to become engaged in problem-solving and excited about solving challenging problems. It offers an opportunity to understand the standards of problem solving and reasoning, and a path to achieving these standards in their classrooms.

Since 1995 we have offered (initially with support from the National Science Foundation) an institute for K-8 teachers, the Leadership Program in Discrete Mathematics (LP-DM). What we have found is that discrete mathematics provides teachers, particularly elementary school teachers, with non-intimidating access to interesting and important mathematical ideas and strategies that they can use in their classrooms to strengthen reasoning and problem-solving skills for students at all levels and of all abilities. Also, because of its visual nature, discrete mathematics appeals to the learning style of many elementary school teachers; many teachers find, for the first time, an area of mathematics that can be presented in a way that fits their learning styles and the learning styles of their students.

In the very first activity of the LP-DM, groups of participants are engaged in coloring maps. They sit around large maps of the United States, where the interiors of all the states are white, and try to determine (using chips of different colors) the minimum number of colors that must be used if you want to color each state and, at the same time, ensure that bordering states are colored using different colors so that the borders are recognizable. All teachers are engaged in this activity, all find it both fun and challenging, all are exercising their problem-solving and reasoning skills, and all realize that the same activity will engage *all* of their students, independent of grade level or ability level. Not only that, but they also realize that *all* of their students can be successful at such problems.

Here is how this realization comes about[10]. Initially, teachers do not recognize that they are working on mathematical problems, so their own negative associations with mathematics are not activated. They find that they are successful at doing these problems. They learn that they were doing mathematics all along. They realize that they can be successful at mathematics, although they always thought that impossible. They conclude that if they can do it, then so can their students. When they saw themselves as unsuccessful at mathematics, it was reasonable to expect that some of their students would also be unsuccessful at mathematics. But now when they realized that they could be successful at mathematics, then their expectations were raised for *all* of their students.

Since most teachers are unfamiliar with discrete mathematics, questions like this are not even posed in traditional math classes, depriving students of a rich source of problem-solving situations. This activity also introduces them to mathematical applications; techniques for coloring maps efficiently are used in solving a variety of scheduling problems, such as class scheduling, where two courses that share a student (i.e., a student is taking both courses) have to be assigned different meeting times.

These kinds of experiences provide teachers, both practicing and prospective, with an understanding of what their students can achieve mathematically, and the tools for them to become effective teachers of mathematics.

Discrete mathematics is more than just a collection of new and interesting mathematical topics: it is also a vehicle for giving teachers a new way to think about

[10] We hope to initiate a study this year that will determine the extent to which the anecdotal account that follows reflects reality.

traditional mathematical topics and a new strategy for engaging their students in the study of mathematics.

Looking to the Future

Although NCTM's *Principles and Standards for School Mathematics* proclaims that discrete mathematics will be incorporated into the discussion of the other standards, the actual references to the topics of discrete mathematics in the document itself are few and far between. Moreover, they are generally accidental – that is, there is no reference to their source in discrete mathematics. The authors of the standards, unfortunately, seem unaware of discrete mathematics and the opportunities discrete mathematics provides to implement the standards.

Over time, this will change. Indeed, paraphrasing the concluding sentence of the abstract of this article, "more educators will learn these topics and will recognize their value in achieving the goal of improving students' understanding of mathematics." Accompanying this heightened understanding of the value of discrete mathematics will be a proliferation of research studies that verify the claim that a healthy dose of discrete mathematics, if properly administered, will improve mathematical understanding of teachers (particularly K-8 teachers) and students (at all levels).

"If properly administered …" A few words of caution. Discrete mathematics can be routinized as effectively as any other topic. Memorizing algorithms of discrete mathematics can become as deadly as memorizing arithmetic algorithms, and with less value. An important value of discrete mathematics is that it offers an easy way of accessing the mathematical enterprise of conjecture, exploration, discovery, and proof, and provides students and teachers with "Aha!" opportunities. Focusing on the content of discrete mathematics to the detriment of the spirit of discrete mathematics will be counter-productive, which explains why "if properly administered" appears in the previous paragraph. The key to this is, as always, proper preparation of teachers.

Should there be an AP test in discrete mathematics? Considering the comments that I made earlier about the AP test in calculus, it is not surprising that I would not advocate for an AP test in discrete mathematics. That is not to say that the content of discrete mathematics is unimportant. It is of course important, and those students who are mathematically inclined will benefit from a rigorous study of discrete mathematics. However, in my eyes, the greater value of discrete mathematics is the opportunity it offers to the vast majority of students who would benefit from a glimpse of what mathematics is all about, a glimpse that they will not get from exclusive focus on traditional topics. We must keep in mind that discrete mathematics is an important vehicle for democratizing mathematics.

Conclusion

In this short article, I have described the nature of discrete mathematics and its dual value in K-12 education – as a collection of important content material and as a vehicle for improving mathematics education. I have indicated that discrete mathematics can be valuable to students of all levels of ability and achievement, and have described the important role it can play in opening a door to mathematics to elementary school teachers. Over time, more people will come to know the subject and recognize its value; over time, I anticipate that the claims that I am making in this article based on our experience will also be supported by research studies. I hope that this process will be accelerated as a result of the publication of the following two sets of materials in the coming years – the NCTM's volume on discrete mathematics in its Navigations Series (*Navigating through Discrete Mathematics*, authored by Valerie DeBellis, Eric Hart, Margaret Kenney, and myself), and a textbook for a mathematics course for prospective K-8 teachers (*Making Mathematics Engaging: Discrete Mathematics for K-8 Teachers*, authored by Valerie DeBellis and myself).

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Figure 1. What is the shortest path from A to B?