New Jersey's *Mathematics Standards*

Two Vignettes

Somewhere in a New Jersey ***elementary school:***

The students in Mrs. Chaplain's fifth grade class eagerly return from recess, excited by the prospect of working on another of her famous Chaplain's Challenges. Mrs. Chaplain regularly uses a Challenge with her math class, and today she has promised the children that the problem would be a great one. She believes that all of her students will be up to the Challenge and expects that it will engage them in an exploration and discussion of the relationship between area and perimeter.

> *Suppose you had 64 meters of fencing with which you were going to build a pen for your large pet dog. What are some of the different pens you could build if you used all of the fencing? Which pen would have the most play space? Which would give the most running space? What would be the* ***best*** *pen?* *

As the students file into the classroom, they stand, scattered around the room, reading the Challenge from the board even before finding their seats. Then they begin to ask questions about it. *What is fencing? Wouldn't the pen with the most play space be the same as the one with the most running space? What shapes are allowed?* Mrs. Chaplain answers some of these questions directly (she has brought a sample of fencing to class so she could show the students what it looks like), but, for the most part, she tells the students that they can discuss their questions in their regular working groups.

Continued in the left column on the following pages

Somewhere in a New Jersey ***high school:***

Ms. Diego's algebra class and Mr. Browning's physical science class are jointly investigating radioactive decay. The two teachers, with the support of the school administrators, have worked out a schedule that enables their classes to meet together this month to explore some of the mathematical aspects of the physical sciences. Both teachers regularly incorporate some content from the other's discipline in class activities, but this month was specially planned to be a kind of celebration of the relationship between the two areas. By the end of the month, they expect that the students will really appreciate the role that mathematics plays in the sciences, and the problems that are presented by the sciences that call for innovative mathematical solutions.

The classes are average. Nearly every student in the high school takes these two classes at some point during their stay and, over the past few years, because of exciting real-world problems like the one on which they are working this week, the classes have become two of the most popular in the school.

Monday's class begins with a presentation by Mr. Browning about the process of carbon dating. He describes the problem that archaeologists faced in the 1940s with respect to determining the age of a fossil. They knew that all living things contained a predictable amount of radioactive carbon that began to diminish as soon as the organism died. If they could measure the amount that remained in some discovered fossil and if they knew the rate

Continued in the right column on the following pages

*This problem was adapted from one that appears in the *Professional Standards for Teaching Mathematics*, National Council of Teachers of Mathematics, 1991.

In the elementary school ...	*In the high school ...*
The groups begin their exploration by discussing how to organize their efforts. One of the first questions to arise is what kinds of tools would help solve the problem. The students have used a variety of materials to deal with Chaplain's Challenges, and they have often found that the groups that fashioned the best models of the problem situations were the ones that found it easiest to find solutions. Today, one group decides to use the 10-by-10 geoboards, figuring that they can quickly make a lot of different "pens" out of rubber bands if they let the space between nails equal four meters. Another group decides to get some graph paper on which to draw their pens, because that gives them a lot of flexibility. Still another group, reluctant to be limited to rectangular shapes and work spaces, thinks that the geometry construction software loaded on the computers in the back of the room would let them draw a variety of shapes and even help them measure various characteristics of the shapes. One last group, striving for realism, decides to use a loop of string sixty-four inches long. As the session progresses, the groups of students make many sample pens with whatever materials they have chosen to use. Some groups switch materials as they perceive other materials to be less restrictive than the ones they are using. Keeping the perimeter a constant 64 meters, they measure the areas of the pens using some of the strategies they developed the week before. Mrs. Chaplain circulates around the room, paying careful attention to the contributions of individual students, making notes to herself about two particular children, one who seems to be having difficulty with the concept of area, and another who is doing a nice job of leading her group to a solution. Gradually, the work becomes more symbolic and verbal and less concrete. The students begin to make tables to record the dimensions and descriptions of their pens and to look for some kind of pattern, because they have learned from experience that this frequently leads to insights.	at which the carbon "decayed," they could figure out the age of the object. An American chemist named Willard Libby developed a technique that allowed them to do so. Ms. Diego explains that the classes will spend the next few days exploring the concept of radioactive decay and, toward the end of the week, they will be able to solve some of the same kinds of problems solved by those archaeologists. On Tuesday, working at stations created by the teachers, the students begin to explore both the mathematical and scientific aspects of radioactive decay. Working in groups, the students use sets of 50 dice to simulate collections of radioactive nuclei. Each roll of the collection of dice represents the passage of one day. Any time a die lands with a "1" showing, it "spontaneously decays" and is taken out of the collection. The students plot the number of radioactive nuclei left versus the number of days passed in an effort to determine the half-life of the element — the amount of time it takes for half of the element to decay. Because the experiment is relatively well controlled, each group working on the task produces a graph that effectively illustrates the decay, but, because the process is also a truly random process, each group's results are slightly different from those of other groups. On Wednesday, in a very different kind of activity, students use graphing calculators in a guided activity to discover properties of exponential functions, and the effects on the graphs of various changes to the parameters in the functions. Working from a worksheet prepared by the teachers, they start with the general form of an exponential function, $y = ab^x$. Using the values $a = 1$ and $b = 2$, they input the equation into the calculators and study the resulting graph. Then, they systematically change the values of a and b to discover what each change does to the graph. They are directed by the worksheet to pay particular attention to the effect of changing b to a value between zero and one, because graphs of that t t t

In the elementary school ...

One group follows the teacher's suggestion and enters their table of values for rectangular pens into a computer, generating a broken-line graph of the length of the pen versus its area.

Toward the end of the class, the students become comfortable with their discoveries. Mrs. Chaplain reflects again on how glad she is that the faculty decided to organize the school schedule in such a way as to allow for these extended class sessions. When she sees how involved and active the students are, how they try to persuade each other to follow one path or another, how their verbalizations either cement their own understandings or provide opportunities for others to point out flaws in their thinking, she realizes that only with this kind of time and this kind of effort can she do an adequate job of teaching mathematics.

The summary discussion at the end of the session allows the students an opportunity to see what their classmates have done and to evaluate their own group's results. Mrs. Chaplain learns that everyone in the class understands that if you hold the perimeter constant, you can create figures with a whole range of areas. Moreover, she feels that a majority of the class also has come to the generalization that the more compact a figure is, the greater its area, and the more stretched out it is, the smaller its area.

But the students still have very different answers to the question, *What would be the* ***best*** *pen?* That fits her plans perfectly. For homework, Mrs. Chaplain asks each student to design the pen that he or she thinks is best, draw a diagram of it, label its dimensions and its area, and write a paragraph about why that particular pen would be best for the dog. Mrs. Chaplain plans to move on from this activity to others where the students concentrate on more efficient strategies for finding the areas of some of the non-rectangular shapes they explored in this Challenge.

In the high school ...

that type will be especially important for their work with radioactive decay. The culminating problem on the worksheet is a challenge to try to find the values of a and b that produce a graph that looks like the ones that resulted from the experiment with the dice. The students enjoy the problem and use their calculators to quickly check and refine their solutions, zeroing in on the critical numbers. There is a lot of discussion about why those numbers might be the correct ones.

On Thursday, the students discuss a reading that was assigned for homework the night before, focusing on carbon dating and addressing some of the mathematical processes used to determine the age of fossils. This discussion is led by the two teachers, who have brought in some fossilized samples to better acquaint the students with the kind of materials they read about. Ms. Diego then leads a session to develop the computational procedures for solving the carbon dating problems using exponential functions. The students will be given some homework problems of this type and will spend tomorrow's class discussing those problems and wrapping up the unit.

The teachers are very pleased with what the classes have accomplished. The active involvement with a hands-on experiment simulating decay, the symbolic manipulations and graph explorations made possible by the graphing calculator, and the study of a particular scientific application of the mathematics have been very productive. By working together as a team, the teachers have been able to relate the different aspects of the phenomenon to each other. The students have learned a great deal of both mathematics and science and have seen how strongly they are linked.

The vision of the *Mathematics Standards* of the New Jersey State Department of Education's *Core Curriculum Content Standards* revolves around what takes place in classrooms like those described in the previous pages. It is focused on achieving one crucial goal:

GOAL: To enable ALL of New Jersey's children to move into the twenty-first century with the mathematical skills, understandings, and attitudes that they will need to be successful in their careers and daily lives.

As more and more New Jersey teachers incorporate the recommendations of the *Mathematics Standards* into their teaching, we should be able to see the following results (as described in *Mathematics to Prepare Our Children for the 21st Century: A Guide for New Jersey Parents*, published by the New Jersey Mathematics Coalition in September 1994).

Students who are excited by and interested in their activities. A principal goal is for children to learn to enjoy mathematics. Students who are excited by what they are doing are more likely to truly understand the material, to stay involved over a longer period of time, and to take more advanced courses voluntarily. When math is taught with a problem-solving spirit, and when children are allowed to make their own hands-on mathematical discoveries, math can be engaging for all students.

Students who are learning important mathematical concepts rather than simply memorizing and practicing procedures. Student learning should be focused on understanding when and how mathematics is used and how to apply mathematical concepts. With the availability of technology, students need no longer spend the same amount of study time practicing lengthy computational processes. More effort should be devoted to the development of number sense, spatial sense, and estimation skills.

Students who are posing and solving meaningful problems. When students are challenged to use mathematics in meaningful ways, they develop their reasoning and problem-solving skills and come to realize the potential usefulness of mathematics in their lives.

Students who are working together to learn mathematics. Children learn mathematics well in cooperative settings, where they can share ideas and approaches with their classmates.

Students who write and talk about math topics every day. Putting thoughts into words helps to clarify and solidify thinking. By sharing their mathematical understandings in written and oral form with their classmates, teachers, and parents, students develop confidence in themselves as mathematical learners; this practice also enables teachers to better monitor student progress.

Calculators and computers being used as important tools of learning. Technology can be used to aid teaching and learning, as new concepts are presented through explorations with calculators or computers. But technology can also be used to assist students in solving problems, as it is used by adults in our society. Students should have access to these tools, both in school and after school, whenever they can use technology to do more powerful mathematics than they would otherwise be able to do.

Teachers who have high expectations for ALL of their students. This vision includes a set of achievable, high-level expectations for the mathematical understanding and performance of all students. Although more ambitious than current expectations for most students, these standards are absolutely essential if we are to reach our goal. Those students who can achieve more than this set of expectations must be afforded the opportunity and encouraged to do so.

A variety of assessment strategies rather than sole reliance on traditional short-answer tests. Strategies including open-ended problems, teacher interviews, portfolios of best work, and projects, in combination with traditional methods, will provide a more complete picture of students' performance and progress.

Learning environments like this **should and can become the reality in virtually all New Jersey classrooms before the turn of the century**. Making this vision a reality is both necessary and achievable.

The Necessity of the Vision

Perhaps the most compelling reason for this vision of mathematics education is that our children will be better served by higher expectations, by curricula which go far beyond basic skills and include a variety of mathematical models, and by programs which devote a greater percentage of instructional time to problem-solving and active learning. Many students respond to the current curriculum with boredom and discouragement, develop the perception that success in mathematics depends on some innate ability which they simply do not have, and feel that, in any case, mathematics will never be useful in their lives. Learning environments like the one described in the vision will help students to enjoy and appreciate the value of mathematics, to develop the tools they need for varied educational and career options, and to function effectively as citizens and consumers.

Preparing our students for careers in the twenty-first century also requires that we make this vision a reality. Our curricula are often preoccupied with what national reports call "shopkeeper mathematics," competency in the basic operations that were needed to run a small store several generations ago; yet very few of our students will have careers as shopkeepers. To compete in today's global, information-based economy, students must also be able to solve real problems, reason effectively, and make logical connections. Jobs requiring mathematical knowledge and skills in areas such as data analysis, problem-solving, pattern recognition, statistics, and probability are growing at nearly twice the rate of growth of overall employment. To prepare students for such careers, the mathematics curriculum must change.

We must take seriously the goal of preparing *all* students for twenty-first century careers. In order to do this, we must overcome the all too common perception among students that they simply lack mathematical ability. *Everybody Counts*, a 1989 report prepared by the Mathematical Sciences Education Board of the National Academy of Sciences, notes the following:

> Only in the United States do people believe that learning mathematics depends on special ability. In other countries, students, parents, and teachers all expect that most students can master mathematics if only they work hard enough. The record of accomplishment in these countries — and in some intervention programs in the United States — shows that most students can learn much more mathematics than is commonly assumed in this country (MSEB, 1989, 10).

Curricula that assume student failure are bound to fail; we need to develop curricula that assume student success.

Not only will our students need to find employment in the twenty-first century, but our state and country will need to find employees. American schools have done well in the past at producing a relatively small mathematical elite that adequately served the needs of an industrial/mechanical economy. But that level of "production" is no longer good enough. The global economy in which graduates of our schools will seek employment is more competitive than ever and is rapidly changing in response to advances in technology. Products and factories are being designed by mathematical models and computer simulations, computers are controlling production processes and plants, and robots are replacing workers on assembly lines. Our state and our country need people with the skills to develop and manage these new technologies. In the past, industry moved in search of cheap labor; today, industry frequently moves in search of skilled labor. Our unemployment problem is not only one of too few jobs, but also one of too few skilled workers for existing jobs. We must not only strive to provide our graduates with the skills for 21st century jobs, but also to ensure that the number of graduates with those skills is sufficient for the needs of our state and our nation.

Toward Achieving the Vision

To achieve the vision, the first step is to translate it into specific goals. That is the purpose of the *Mathematics Standards*. The term *standards* as used here encompasses both *goals* and *expectations,* but it also is meant to convey the older meaning of *standards*, a *banner*, or a *rallying point*. These mathematics standards are intended to be a definition of excellent practice, and a description of what can be achieved if all New Jersey communities rally behind the standards, so that this excellent practice becomes common practice.

This vision of excellent mathematical education is based on the twin premises that *all* students *can* learn mathematics and that all students *need* to learn mathematics. Therefore, for all of the reasons mentioned previously, it is essential that we offer students the very highest quality of mathematics education possible. The *Mathematics Standards* were not designed as minimum standards, but rather as world-class standards which will enable all of our students to compete in the global marketplace of the 21st century.

Sixteen mathematics core curriculum content standards, describing what students should know and be able to do, have been adopted by the New Jersey State Board of Education. Two additional standards explicitly address how the learning environment in classrooms can foster success in mathematics for all students and can link assessment to learning and instruction.

These eighteen standards define the critical goals of mathematical education today. In addition to more familiar content, there are many topics which have not been part of the traditional curriculum. Included also are new emphases on the whys and hows of mathematics learning: posing and solving real world problems, effectively communicating mathematical ideas, making connections within mathematics and between mathematics and other areas, active student involvement, the uses of technology, and the relationship between assessment and instruction.

In the future these standards may undergo revision. The standards must be dynamic, and we must be prepared to revise them with changes in mathematics and its use.

Overview of New Jersey's *Mathematics Standards*

Background

A draft version of the *New Jersey Mathematics Standards* was developed by a panel of thirty-one individuals who met extensively during the 1992-1993 school year. Crafted by a broad range of New Jersey elementary, middle school, and secondary teachers, supervisors, administrators, college mathematics educators, mathematicians, and representatives of business and industry, the draft *New Jersey Mathematics Standards* was intended to provide a clear vision of exemplary mathematics learning and to define and then articulate the standards necessary for achieving quality mathematics education.

After the completion of the draft *New Jersey Mathematics Standards*, over 7000 copies of the document were distributed for review across the state. At the same time, efforts began to extend the draft *New Jersey Mathematics Standards* into a mathematics framework. The *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework*, published in March 1995, contained a revised version of the standards which addressed many of the comments of both the reviewers of the draft standards and the drafters of the framework materials. As a result of this process, the standards in the *Preliminary Version* represented a statewide consensus of what mathematics educators believe are high achievable goals for all students.

During 1995, a new working group built upon these draft standards and, together with similar working groups in other content areas, engaged the public in an extensive review process that resulted in modest modifications of the draft standards in mathematics. This process culminated in the adoption on May 1, 1996 by the New Jersey State Board of Education of the *Core Curriculum Content Standards*, which includes the *Mathematics Standards*, standards in six other content areas, and cross-content workplace readiness standards.

Building on the National Standards

Although philosophically aligned with the *Curriculum and Evaluation Standards for School Mathematics* of the National Council of Teachers of Mathematics (NCTM, 1989), New Jersey's *Mathematics Standards* are designed to reflect conditions specific to New Jersey, as well as national changes in mathematics education since the NCTM document was written.

New Jersey is a state on the forefront of industrial and academic uses of technology and the national leader in numerous scientific industries. Our work force, it could be argued, has a greater need for mathematical and scientific fluency than that of any other state in the country. At the same time, the state is highly urbanized, has a tremendously diverse population, and presently delivers its educational programs through a network of more than 600 independent school districts. These educational and demographic characteristics present a truly unique setting in which to establish standards.

The standards rest on the notion that an appropriate mathematics curriculum results from a series of critical decisions about three inseparably linked components: content, instruction, and assessment. The standards will only promote substantial and systemic improvement in mathematics education if the *what* of the content being learned, the *how* of the problem-solving orientation, and the *where* of the active, equitable, involving learning environment are synergistically woven together in every classroom.

The mathematical environment of every child must be rich and complex and all students must be afforded the opportunity to develop an understanding and a command of mathematics in an environment that provides for both affective and intellectual growth. Particular to New Jersey's *Mathematics Standards* is the definition of an appropriate mathematical learning environment.

New Jersey's *Mathematics Standards* also contain a strong focus on the use of technology as a regular, integral part of school mathematics curricula at every grade level. The state mandate for the use of calculators on statewide assessment is but one indication of the strong movement that has already begun in this direction. Teachers and students who adopt these standards will understand, and develop the abilities to use, powerful, up-to-date mathematics and technology.

Although ours is a geographically small state, it has a widely diverse population. Children enter our schools from a tremendous variety of backgrounds and cultures. One of the roles of New Jersey's *Mathematics Standards*, therefore, is to specify a set of achievable high-level expectations for the mathematical understanding and performance of *all* students. The expectations included in the standards are substantially more ambitious than current expectations for most students, but we believe that they are attainable by all students in the state. Those New Jersey students who can achieve more than this set of expectations must be afforded the opportunity and encouraged to do so.

A Core Curriculum for Grades K-12

Implicit in the vision and standards is the notion that there should be a core curriculum for grades K-12. What does a "core curriculum" mean? It means that every student will be involved in experiences addressing all of the expectations of each of the sixteen content standards. It also means that all courses of study should have a common goal of completing this core curriculum, no matter how students are grouped or separated by needs and/or interests.

A core curriculum does not mean that all students will be enrolled in the same courses. Students have different aptitudes, interests, educational and professional plans, learning habits, and learning styles. Different groups of students should address the core curriculum at different levels of depth, and should complete the core curriculum according to different timetables. Nevertheless, all students should complete all elements of the core curriculum recommended in the *Mathematics Standards*.

All students should be challenged to reach their maximum potential. For many students, the core curriculum described here will indeed be challenging. But if we do not provide this challenge, we will be doing our students a great disservice — leaving them unprepared for the technological and information age of the 21st century.

For other students, this core curriculum itself will not be a challenge. We have to make sure that we provide these students with appropriate mathematical challenges. We have to make sure that the raised expectations for all students do not result in lowered expectations for our high achieving students. A core curriculum does not exclude a program which challenges students beyond the expectations set in the *Mathematics Standards.* Indeed, the *Mathematics Standards* call for all schools to provide opportunities to their students to learn more mathematics than is contained in the core curriculum.

The issue of a core curriculum, and its implications, is discussed at greater length in the chapter on Standard 16.

New Jersey's *Mathematics Standards*

The *Mathematics Standards* consist of eighteen statements which describe what is essential to excellent mathematics education and presents a view of mathematics teaching and learning that integrates the processes of mathematical activity, the content of the mathematics, and the learning environment in the classroom. The following sixteen standards were adopted by the New Jersey State Board of Education.

1. All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.
2. All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.
3. All students will connect mathematics to other learning by understanding the interrelationships of mathematical ideas and the roles that mathematics and mathematical modeling play in other disciplines and in life.
4. All students will develop reasoning ability and will become self-reliant, independent mathematical thinkers.
5. All students will regularly and routinely use calculators, computers, manipulatives, and other mathematical tools to enhance mathematical thinking, understanding, and power.
6. All students will develop number sense and an ability to represent numbers in a variety of forms and use numbers in diverse situations.
7. All students will develop spatial sense and an ability to use geometric properties and relationships to solve problems in mathematics and in everyday life.
8. All students will understand, select, and apply various methods of performing numerical operations.
9. All students will develop an understanding of and will use measurement to describe and analyze phenomena.
10. All students will use a variety of estimation strategies and recognize situations in which estimation is appropriate.
11. All students will develop an understanding of patterns, relationships, and functions and will use them to represent and explain real-world phenomena.
12. All students will develop an understanding of statistics and probability and will use them to describe sets of data, model situations, and support appropriate inferences and arguments.
13. All students will develop an understanding of algebraic concepts and processes and will use them to represent and analyze relationships among variable quantities and to solve problems.
14. All students will apply the concepts and methods of discrete mathematics to model and explore a variety of practical situations.
15. All students will develop an understanding of the conceptual building blocks of calculus and will use them to model and analyze natural phenomena.
16. All students will demonstrate high levels of mathematical thought through experiences which extend beyond traditional computation, algebra, and geometry.

In addition, the New Jersey State Department of Education's *Core Curriculum Content Standards* includes, in its Introduction to the *Mathematics Standards*, the following two "learning environment standards."

17. All students' mathematical learning will embody the concepts that engagement in mathematics is essential, and that decision-making, risk-taking, cooperative work, perseverance, self-assessment, and self-confidence are frequently keys to success.
18. All students will be evaluated by using a diversity of assessment tools and strategies, to provide multiple indicators of the quality of every student's mathematical learning and of overall program effectiveness.

Descriptive Statements and Cumulative Progress Indicators

Accompanying each of the *Mathematics Standards* is a "descriptive statement" and "cumulative progress indicators," which together provide a brief elaboration of the standard. The descriptive statement expands the simple statement of the standard into a paragraph which outlines the meaning and significance of the standard. The cumulative progress indicators describe what students who are working to achieve the standard should understand and be able to do at each of grades 4, 8, and 12. These materials begin on page 15.

Vignettes

Following the descriptive statements and cumulative progress indicators are a series of short vignettes which suggest how the standards can be effectively integrated in classroom settings. The vignettes are intended to make the standards user-friendly; they serve as examples, as illustrations, of how individual educators can incorporate the standards into their classroom instruction.

Implementing New Jersey's *Mathematics Standards...*

This set of standards is not an end in itself. It represents, instead, a beginning — the beginning of a process intended to mobilize all segments of the education community and the state at large to truly reshape our approach to mathematics education, to achieve the vision.

... Through Statewide Efforts

Statewide efforts to implement the New Jersey's *Mathematics Standards* are proceeding in a number of areas, including curriculum framework, assessment, professional development, and public information.

New Jersey Mathematics Curriculum Framework

An important step in implementing the *Mathematics Standards* is the development of this document, the *New Jersey Mathematics Curriculum Framework*, with the support of a three-year grant from the United States Department of Education. In accepting the grant, the New Jersey Department of Education and the New Jersey Mathematics Coalition also accepted the challenge to develop and implement a world-class curriculum framework which will serve as a model for other states.

The *New Jersey Mathematics Curriculum Framework* contains chapters dealing with each of the standards. Each chapter provides overviews of what the standard means at each of five grade level clusters (K-2, 3-4, 5-6, 7-8, 9-12), and activities which illustrate how the cumulative progress indicators can be achieved at each of those grade levels. The *Framework* will provide assistance and guidance to districts and teachers in how to implement these standards, in translating the vision into reality.

Statewide Assessments

In order for these standards to be implemented, our statewide assessment program must be based on the *Mathematical Standards*; and if these standards truly represent what we value in the learning of mathematics, then that must be reflected in what we assess and how we assess it. The New Jersey Department of Education will continue to develop a statewide assessment program which reflects the *Mathematics Standards*. A fourth-grade statewide mathematics assessment aligned with these standards is now being developed, called the Elementary School Proficiency Assessment (ESPA), and should replace the kinds of standardized tests currently in use which tend to reinforce a traditional low-level, drill-based curriculum. The Eleventh-Grade High School Proficiency Test (HSPT) and the Eighth-Grade Early Warning Test (EWT) will continue to evolve to reflect the *Mathematical Standards*.

Professional Development

In order for the *Mathematics Standards* to be implemented, there must also be a concerted effort to provide professional development activities to enable teachers to achieve in their classrooms the vision described in this document. Teachers at all grade levels will need to understand and utilize new content material, new orientations toward problem-solving and reasoning, and new strategies for helping all students achieve success. To do this, they will need extensive assistance, through expanded opportunities for professional development throughout the state, and commitment and encouragement from their schools and districts to take advantage of those opportunities.

Public Information

At the level of public information, there will be a concerted effort to inform parents and the public about New Jersey's *Mathematics Standards*, and to enlist their cooperation and advocacy in the implementation of the standards. The New Jersey Mathematics Coalition developed and distributed in 1995 a booklet, *Mathematics To Prepare Our Children for The 21st Century: A Guide for New Jersey Parents*. This booklet conveyed the vision and the substance of the standards to the parents of the state, and encouraged them to support the direction and efforts represented by the national and state standards. A revised guide for parents will be published and disseminated widely. Moreover, the Coalition will continue to develop other vehicles for conveying the message of the standards to New Jersey parents and the public, such as activities for parents during Math, Science and Technology Month (MSTM) each April and presentations to parent organizations, and will continue to provide parents with mathematical experiences that reflect the vision of New Jersey's *Mathematics Standards*.

... Through Local Efforts

The *New Jersey Mathematics Curriculum Framework* is designed to provide assistance and guidance to schools and districts in their efforts to implement the *Mathematics Standards*. However, the vision is not something that can be achieved overnight; there is no "magic wand" which will suddenly transform a classroom or a curriculum into one which implements these standards. A decision by a school or a district to work toward achieving the vision involves an ongoing commitment to a process of change. That process should begin now.

Chapter 20 of this *Framework* provides a model for understanding systemic change, and describes specific processes to follow in order to successfully bring about change. Key to the success of efforts designed to bring about systemic change is enlisting the involvement and support of all those affected by the change.

An important first step in each school is to encourage teachers to review and explore together the *Mathematics Standards* and the *New Jersey Mathematics Curriculum Framework*. Each chapter of the *Framework* can serve as a basis for extended discussions involving teachers and administrators, and school personnel are encouraged to form discussion groups for this purpose. They might begin by:

- reviewing together the *Mathematics Standards*, discussing what each standard means, the extent to which they are already addressing each standard, and what next steps they can take;
- selecting individual content chapters from the *Framework* for more extensive review, discussion, and implementation;
- reviewing together the vignettes at the end of this chapter and discussing how their own classroom practices can reflect the diversity of strategies described there;
- using the chapters on the two learning environment standards as a basis for review of their instructional and assessment techniques; and
- developing their own recommendations concerning how the school or district can begin its efforts to achieve the vision.

However, teachers cannot carry out these suggestions without support. To facilitate this process, we encourage schools and districts to:

- create opportunities for teachers to meet together regularly, possibly through the scheduling of common planning times;
- actively encourage teachers in their explorations, providing resources to support such activities;
- take steps to provide more flexible scheduling, permitting extended periods for exploration and contiguous periods for collaboration among mathematics and science teachers; and
- make meaningful professional development activities for teachers an important priority.

All of these activities will be valuable. However, to realize the vision throughout the state, virtually all elements in our educational system must be rethought. Some of the areas of concern and questions which arise are these:

Mathematical Disposition: How can a community of mathematics learners best be created and then fostered in a school setting? How can the positive affective characteristics that we hope for in students be extended to and reinforced by their parents and community?

Equity: What teaching and administrative strategies result in the inclusion of all students in mathematical activities? How can we plan for and achieve true equity in our mathematics classrooms?

Instruction and Assessment: What are the most effective strategies for assuring the adequate integration of curriculum, instruction, and assessment? How can we best assure that appropriate connections are made among mathematical topics and between mathematics and other disciplines, that technology truly becomes a tool for mathematics learning, that mathematics learning is active, involving, and exciting? What materials and resources are necessary to assure our success? What means of evaluation will best allow us to measure our progress toward these goals?

Professional Development: What are the most effective strategies for preparing teachers at both the pre-service and in-service levels to teach in a manner consistent with these standards? What efforts are necessary to develop and nurture the cadre of mathematics teacher leaders that will be needed to move the vision beyond this document? What is the role of the state's college and university faculties in this process?

School Organization: What type of school culture and ethos must be in place before these recommended changes in orientation can begin to take hold and then grow and flourish? What changes are needed in school scheduling and time allocation practices to promote the kind of teaching and learning envisioned here? What implications for staffing and teacher assignment are inherent in the standards?

Educational Policy: What changes in the state administrative code or the body of state mandates would further encourage the reform suggested here? What local school district policies either inhibit or promote these efforts? What are the roles of local school administrators and school board members in support of the standards? How can this document be used as a vehicle for change? What mechanisms are in place to assure that the vision it embodies changes and grows with time?

These questions are raised in the *New Jersey Mathematics Curriculum Framework*, and should be part of the ongoing discussion at the local level, as well as at the state level, of how the *Mathematics Standards* can best be implemented in New Jersey schools.

Summary

The *Mathematics Standards* presented in this chapter offer a powerful challenge to all teachers, all schools, and all districts in New Jersey — to enable all of our students to step forward into the next century with the mathematical skills, understandings, and attitudes that they will need to be successful in their careers and daily lives.

The *Mathematics Standards* also offer a powerful tool to help us meet that challenge, providing a vision and standards which both inform us and rally us in our efforts.

It will not be easy to meet this challenge, nor can it happen overnight. But it can only happen if all of us together decide to make it happen. There are many obstacles, but we must not let our awareness of the obstacles become yet another obstacle. Let us work together to make the vision of New Jersey's *Mathematics Standards* a reality by the end of this century!

References

Mathematical Sciences Education Board. *Everybody Counts*. Washington, DC: National Academy Press, 1989.

National Council of Teachers of Mathematics. *Assessment Standards for School Mathematics*. Reston, VA, 1995.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA, 1989.

National Council of Teachers of Mathematics. *Professional Standards for Teaching Mathematics*. Reston, VA, 1991.

New Jersey State Department of Education. *Core Curriculum Content Standards*. Trenton, NJ, 1996.

New Jersey Mathematics Coalition. *Mathematics to Prepare Our Children for the 21st Century: A Guide for New Jersey Parents*, 1994.

New Jersey Mathematics Coalition. *Preliminary Version* of the *New Jersey Mathematics Curriculum Framework*, 1995.

On-Line Resources

http://dimacs.rutgers.edu/nj_math_coalition/framework.html/

The *Framework* will be available at this site during Spring 1997. In time, we hope to post additional resources relating to this standard, such as grade-specific activities submitted by New Jersey teachers, and to provide a forum to discuss the *Mathematics Standards*.

MATHEMATICS STANDARDS*

Descriptive Statements and Cumulative Progress Indicators

STANDARD 1*	All students will develop the ability to pose and solve mathematical problems in mathematics, other disciplines, and everyday experiences.

Descriptive Statement

Problem posing and problem solving involve examining situations that arise in mathematics and other disciplines and in common experiences, describing these situations mathematically, formulating appropriate mathematical questions, and using a variety of strategies to find solutions. By developing their problem-solving skills, students will come to realize the potential usefulness of mathematics in their lives.

Cumulative Progress Indicators

By the end of **Grade 4,** students:

1. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to early elementary grades.
2. Recognize, formulate, and solve problems arising from mathematical situations and everyday experiences.
3. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations.
4. Pose, explore, and solve a variety of problems, including non-routine problems and open-ended problems with several solutions and/or solution strategies.
5. Construct, explain, justify, and apply a variety of problem-solving strategies in both cooperative and independent learning environments.
6. Verify the correctness and reasonableness of results and interpret them in the context of the problems being solved.
7. Know when to select and how to use grade-appropriate mathematical tools and methods (including manipulatives, calculators and computers, as well as mental math and paper-and-pencil techniques) as a natural and routine part of the problem-solving process.
8. Determine, collect, organize, and analyze data needed to solve problems.

---

*Note that in the *Core Curriculum Content Standards* of the New Jersey State Department of Education, the *Mathematics Standards* are numbered 4.1, 4.2, 4.3, etc., since they are preceded by standards in three other content areas.

9. Recognize that there may be multiple ways to solve a problem.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 4, 5, 6, 7, and 8 above, by the end of **Grade 8**, students:

10. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand mathematical content appropriate to the middle grades.
11. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, and applications to other disciplines.
12. Construct and use concrete, pictorial, symbolic, and graphical models to represent problem situations and effectively apply processes of mathematical modeling in mathematics and other areas.
13. Recognize that there may be multiple ways to solve a problem, weigh their relative merits, and select and use appropriate problem-solving strategies.
14. Persevere in developing alternative problem-solving strategies if initially selected approaches do not work.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 4, 5, 6, 7, 8, 12, and 14 above, by the end of **Grade 12**, students:

15. Use discovery-oriented, inquiry-based, and problem-centered approaches to investigate and understand the mathematical content appropriate to the high school grades.
16. Recognize, formulate, and solve problems arising from mathematical situations, everyday experiences, applications to other disciplines, and career applications.
17. Monitor their own progress toward problem solutions.
18. Explore the validity and efficiency of various problem-posing and problem-solving strategies, and develop alternative strategies and generalizations as needed.

STANDARD 2 All students will communicate mathematically through written, oral, symbolic, and visual forms of expression.

Descriptive Statement

Communication of mathematical ideas will help students clarify and solidify their understanding of mathematics. By sharing their mathematical understandings in written and oral form with their classmates, teachers, and parents, students develop confidence in themselves as mathematics learners and enable teachers to better monitor their progress.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Discuss, listen, represent, read, and write as vital activities in their learning and use of mathematics.
2. Identify and explain key mathematical concepts, and model situations using oral, written, concrete, pictorial, and graphical methods.
3. Represent and communicate mathematical ideas through the use of learning tools such as calculators, computers, and manipulatives.
4. Engage in mathematical brainstorming and discussions by asking questions, making conjectures, and suggesting strategies for solving problems.
5. Explain their own mathematical work to others, and justify their reasoning and conclusions.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 4, and 5 above, by the end of **Grade 8,** students:

6. Identify and explain key mathematical concepts and model situations using geometric and algebraic methods.
7. Use mathematical language and symbols to represent problem situations, and recognize the economy and power of mathematical symbolism and its role in the development of mathematics.
8. Analyze, evaluate, and explain mathematical arguments and conclusions presented by others.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 4, 5, 6, 7, and 8 above, by the end of **Grade 12,** students:

9. Formulate questions, conjectures, and generalizations about data, information, and problem situations.
10. Reflect on and clarify their thinking so as to present convincing arguments for their conclusions.

STANDARD 3 All students will connect mathematics to other learning by understanding the interrelationships of mathematical idea and the roles that mathematics and mathematical modeling play in other disciplines and in life.

Descriptive Statement

Making connections enables students to see relationships between different topics, and to draw on those relationships in future study. This applies within mathematics, so that students can translate readily between fractions and decimals, or between algebra and geometry; to other content areas, so that students understand how mathematics is used in the sciences, the social sciences, and the arts; and to the everyday world, so that students can connect school mathematics to daily life.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. View mathematics as an integrated whole rather than as a series of disconnected topics and rules.
2. Relate mathematical procedures to their underlying concepts.
3. Use models, calculators, and other mathematical tools to demonstrate the connections among various equivalent graphical, concrete, and verbal representations of mathematical concepts.
4. Explore problems and describe and confirm results using various representations.
5. Use one mathematical idea to extend understanding of another.
6. Recognize the connections between mathematics and other disciplines, and apply mathematical thinking and problem solving in those areas.
7. Recognize the role of mathematics in their daily lives and in society.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, and 4 above, by the end of **Grade 8,** students:

8. Recognize and apply unifying concepts and processes which are woven throughout mathematics.
9. Use the process of mathematical modeling in mathematics and other disciplines, and demonstrate understanding of its methodology, strengths, and limitations.
10. Apply mathematics in their daily lives and in career-based contexts.
11. Recognize situations in other disciplines in which mathematical models may be applicable, and apply appropriate models, mathematical reasoning, and problem solving to those situations.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 8, 9, 10 and 11 above, by the end of **Grade 12,** students:

12. Recognize how mathematics responds to the changing needs of society, through the study of the history of mathematics.

STANDARD 4	All student will develop reasoning ability and will become self-reliant, independent mathematical thinkers.

Descriptive Statement

Mathematical reasoning is the critical skill that enables a student to make use of all other mathematical skills. With the development of mathematical reasoning, students recognize that mathematics makes sense and can be understood. They learn how to evaluate situations, select problem-solving strategies, draw logical conclusions, develop and describe solutions, and recognize how those solutions can be applied. Mathematical reasoners are able to reflect on solutions to problems and determine whether or not they make sense. They appreciate the pervasive use and power of reasoning as a part of mathematics.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Make educated guesses and test them for correctness.
2. Draw logical conclusions and make generalizations.
3. Use models, known facts, properties, and relationships to explain their thinking.
4. Justify answers and solution processes in a variety of problems.
5. Analyze mathematical situations by recognizing and using patterns and relationships.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 2, 3, and 5 above, by the end of **Grade 8,** students:

6. Make conjectures based on observation and information, and test mathematical conjectures and arguments.
7. Justify, in clear and organized form, answers and solution processes in a variety of problems.
8. Follow and construct logical arguments, and judge their validity.
9. Recognize and use deductive and inductive reasoning in all areas of mathematics.
10. Utilize mathematical reasoning skills in other disciplines and in their lives.
11. Use reasoning rather than relying on an answer-key to check the correctness of solutions to problems.

Building upon knowledge and skills gained in the preceding grades, and especially demonstrating continued progress in Indicators 2, 5, 8, 9, 10, and 11 above, by the end of **Grade 12**, students:

12. Make conjectures based on observation and information, and test mathematical conjectures, arguments, and proofs.
13. Formulate counter-examples to disprove an argument.

STANDARD 5	All students will regularly and routinely use calculators, computers, manipulatives, and other mathematical tools to enhance mathematical thinking, understanding, and power.

Descriptive Statement

Calculators, computers, manipulatives, and other mathematical tools need to be used by students in both instructional and assessment activities. These tools should be used, not to replace mental math and paper-and-pencil computational skills, but to enhance understanding of mathematics and the power to use mathematics. Historically, people have developed and used manipulatives (such as fingers, base ten blocks, geoboards, and algebra tiles) and mathematical devices (such as protractors, coordinate systems, and calculators) to help them understand and develop mathematics. Students should explore both new and familiar concepts with calculators and computers, but should also become proficient in using technology as it is used by adults, that is, for assistance in solving real-world problems.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Select and use calculators, software, manipulatives, and other tools based on their utility and limitations and on the problem situation.
2. Use physical objects and manipulatives to model problem situations, and to develop and explain mathematical concepts involving number, space, and data.
3. Use a variety of technologies to discover number patterns, demonstrate number sense, and visualize geometric objects and concepts.
4. Use a variety of tools to measure mathematical and physical objects in the world around them.
5. Use technology to gather, analyze, and display mathematical data and information.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 4, and 5 above, by the end of **Grade 8,** students:

6. Use a variety of technologies to evaluate and validate problem solutions, and to investigate the properties of functions and their graphs.

7. Use computer spreadsheets and graphing programs to organize and display quantitative information and to investigate properties of functions.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 1, 2, 3, 5, and 7 above, by the end of **Grade 12,** students:

8. Use calculators and computers effectively and efficiently in applying mathematical concepts and principles to various types of problems.

STANDARD 6	All students will develop number sense and an ability to represent numbers in a variety of forms and use numbers in diverse situations.

Descriptive Statement

Number sense is defined as an intuitive feel for numbers and a common sense approach to using them. It is a comfort with what numbers represent, coming from investigating their characteristics and using them in diverse situations. It involves an understanding of how different types of numbers, such as fractions and decimals, are related to each other, and how they can best be used to describe a particular situation. Number sense is an attribute of all successful users of mathematics.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Use real-life experiences, physical materials, and technology to construct meanings for whole numbers, commonly used fractions, and decimals.
2. Develop an understanding of place value concepts and numeration in relationship to counting and grouping.
3. See patterns in number sequences, and use pattern-based thinking to understand extensions of the number system.
4. Develop a sense of the magnitudes of whole numbers, commonly used fractions, and decimals.
5. Understand the various uses of numbers including counting, measuring, labeling, and indicating location.
6. Count and perform simple computations with money.
7. Use models to relate whole numbers, commonly used fractions, and decimals to each other, and to represent equivalent forms of the same number.
8. Compare and order whole numbers, commonly used fractions, and decimals.
9. Explore real-life settings which give rise to negative numbers.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

10. Understand money notations, count and compute money, and recognize the decimal nature of United States currency.
11. Extend their understanding of the number system by constructing meanings for integers, rational numbers, percents, exponents, roots, absolute values, and numbers represented in scientific notation.
12. Develop number sense necessary for estimation.
13. Expand the sense of magnitudes of different number types to include integers, rational numbers, and roots.
14. Understand and apply ratios, proportions, and percents in a variety of situations.
15. Develop and use order relations for integers and rational numbers.
16. Recognize and describe patterns in both finite and infinite number sequences involving whole numbers, rational numbers, and integers.
17. Develop and apply number theory concepts, such as primes, factors, and multiples, in real-world and mathematical problem situations.
18. Investigate the relationships among fractions, decimals, and percents, and use all of them appropriately.
19. Identify, derive, and compare properties of numbers.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12,** students:

20. Extend their understanding of the number system to include real numbers and an awareness of other number systems.
21. Develop conjectures and informal proofs of properties of number systems and sets of numbers.
22. Extend their intuitive grasp of number relationships, uses, and interpretations, and develop an ability to work with rational and irrational numbers.
23. Explore a variety of infinite sequences and informally evaluate their limits.

STANDARD 7	All students will develop spatial sense and an ability to use geometric properties and relationships to solve problems in mathematics and in everyday life.

Descriptive Statement

Spatial sense is an intuitive feel for shape and space. It involves the concepts of traditional geometry, including an ability to recognize, visualize, represent, and transform geometric shapes. It also involves other, less formal ways of looking at two- and three-dimensional space, such as paper-folding, transformations, tessellations, and projections. Geometry is all around us in art, nature, and the things we make. Students of geometry can apply their spatial sense and knowledge of the properties of shapes and space to the real world.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Explore spatial relationships such as the direction, orientation, and perspectives of objects in space, their relative shapes and sizes, and the relations between objects and their shadows or projections.
2. Explore relationships among shapes, such as congruence, symmetry, similarity, and self-similarity.
3. Explore properties of three- and two-dimensional shapes using concrete objects, drawings, and computer graphics.
4. Use properties of three- and two-dimensional shapes to identify, classify, and describe shapes.
5. Investigate and predict the results of combining, subdividing, and changing shapes.
6. Use tessellations to explore properties of geometric shapes and their relationships to the concepts of area and perimeter.
7. Explore geometric transformations such as rotations (turns), reflections (flips), and translations (slides).
8. Develop the concepts of coordinates and paths, using maps, tables, and grids.
9. Understand the variety of ways in which geometric shapes and objects can be measured.
10. Investigate the occurrence of geometry in nature, art, and other areas.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

11. Relate two-dimensional and three-dimensional geometry using shadows, perspectives, projections and maps.
12. Understand and apply the concepts of symmetry, similarity and congruence.
13. Identify, describe, compare, and classify plane and solid geometric figures.
14. Understand the properties of lines and planes, including parallel and perpendicular lines and planes,

and intersecting lines and planes and their angles of incidence.

15. Explore the relationships among geometric transformations (translations, reflections, rotations, and dilations), tessellations (tilings), and congruence and similarity.
16. Develop, understand, and apply a variety of strategies for determining perimeter, area, surface area, angle measure, and volume.
17. Understand and apply the Pythagorean Theorem.
18. Explore patterns produced by processes of geometric change, relating iteration, approximation, and fractals.
19. Investigate, explore, and describe geometry in nature and real-world applications, using models, manipulatives, and appropriate technology.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 16 and 19 above, by the end of **Grade 12**, students:

20. Understand and apply properties involving angles, parallel lines, and perpendicular lines.
21. Analyze properties of three-dimensional shapes by constructing models and by drawing and interpreting two-dimensional representations of them.
22. Use transformations, coordinates, and vectors to solve problems in Euclidean geometry.
23. Use basic trigonometric ratios to solve problems involving indirect measurement.
24. Solve real-world and mathematical problems using geometric models.
25. Use inductive and deductive reasoning to solve problems and to present reasonable explanations of and justifications for the solutions.
26. Analyze patterns produced by processes of geometric change, and express them in terms of iteration, approximation, limits, self-similarity, and fractals.
27. Explore applications of other geometries in real-world contexts.

STANDARD 8	All students will understand, select, and apply various methods of performing numerical operations.

Descriptive Statement

Numerical operations are an essential part of the mathematics curriculum. Students must be able to select and apply various computational methods, including mental math, estimation, paper-and-pencil techniques, and the use of calculators. Students must understand how to add, subtract, multiply, and divide whole numbers, fractions, and other kinds of numbers. With calculators that perform these operations quickly and accurately, however, the instructional emphasis now should be on understanding the meanings and uses of the operations, and on estimation and mental skills, rather than solely on developing paper-and-pencil skills.

Cumulative Progress Indicators

By the end of **Grade 4,** students:

1. Develop meaning for the four basic arithmetic operations by modeling and discussing a variety of problems.
2. Develop proficiency with and memorize basic number facts using a variety of fact strategies (such as "counting on" and "doubles").
3. Construct, use, and explain procedures for performing whole number calculations in the various methods of computation.
4. Use models to explore operations with fractions and decimals.
5. Use a variety of mental computation and estimation techniques.
6. Select and use appropriate computational methods from mental math, estimation, paper-and-pencil, and calculator methods, and check the reasonableness of results.
7. Understand and use relationships among operations and properties of operations.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicator 6 above, by the end of **Grade 8,** students:

8. Extend their understanding and use of arithmetic operations to fractions, decimals, integers, and rational numbers.
9. Extend their understanding of basic arithmetic operations on whole numbers to include powers and roots.
10. Develop, apply, and explain procedures for computation and estimation with whole numbers, fractions, decimals, integers, and rational numbers.
11. Develop, apply, and explain methods for solving problems involving proportions and percents.
12. Understand and apply the standard algebraic order of operations.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicator 6 above, by the end of **Grade 12**, students:

13. Extend their understanding and use of operations to real numbers and algebraic procedures.
14. Develop, apply, and explain methods for solving problems involving factorials, exponents, and matrices.

STANDARD 9	All students will develop an understanding of and will use measurement to describe and analyze phenomena.

Descriptive Statement

Measurement helps describe our world using numbers. We use numbers to describe simple things like length, weight, and temperature, but also complex things such as pressure, speed, and brightness. An understanding of how we attach numbers to those phenomena, familiarity with common measurement units like inches, liters, and miles per hour, and a practical knowledge of measurement tools and techniques are critical for students' understanding of the world around them.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Use and describe measures of length, distance, capacity, weight, area, volume, time, and temperature.
2. Compare and order objects according to some measurable attribute.
3. Recognize the need for a uniform unit of measure.
4. Develop and use personal referents for standard units of measure (such as the width of a finger to approximate a centimeter).
5. Select and use appropriate standard and non-standard units of measurement to solve real-life problems.
6. Understand and incorporate estimation and repeated measures in measurement activities.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

7. Use estimated and actual measurements to describe and compare phenomena.
8. Read and interpret various scales, including those based on number lines and maps.
9. Determine the degree of accuracy needed in a given situation and choose units accordingly.
10. Understand that all measurements of continuous quantities are approximate.
11. Develop formulas and procedures for solving problems related to measurement.
12. Explore situations involving quantities which cannot be measured directly or conveniently.

13. Convert measurement units from one form to another, and carry out calculations that involve various units of measurement.
14. Understand and apply measurement in their own lives and in other subject areas.
15. Understand and explain the impact of the change of an object's linear dimensions on its perimeter, area, or volume.
16. Apply their knowledge of measurement to the construction of a variety of two- and three-dimensional figures.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12,** students:

17. Use techniques of algebra, geometry, and trigonometry to measure quantities indirectly.
18. Use measurement appropriately in other subject areas and career-based contexts.
19. Choose appropriate techniques and tools to measure quantities in order to achieve specified degrees of precision, accuracy, and error (or tolerance) of measurements.

STANDARD 10	All students will use a variety of estimation strategies and recognize situations in which estimation is appropriate.

Descriptive Statement

Estimation is a process that is used constantly by mathematically capable adults, and that can be mastered easily by children. It involves an educated guess about a quantity or a measure, or an intelligent prediction of the outcome of a computation. The growing use of calculators makes it more important than ever that students know when a computed answer is reasonable; the best way to make that decision is through estimation. Equally important is an awareness of the many situations in which an approximate answer is as good as, or even preferable to, an exact answer.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Judge without counting whether a set of objects has less than, more than, or the same number of objects as a reference set.
2. Use personal referents, such as the width of a finger as one centimeter, for estimations with measurement.
3. Visually estimate length, area, volume, or angle measure.
4. Explore, construct, and use a variety of estimation strategies.
5. Recognize when estimation is appropriate, and understand the usefulness of an estimate as distinct

from an exact answer.

6. Determine the reasonableness of an answer by estimating the result of operations.
7. Apply estimation in working with quantities, measurement, time, computation, and problem solving.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicators 5 and 6 above, by the end of **Grade 8,** students:

8. Develop, apply, and explain a variety of different estimation strategies in problem situations involving quantities and measurement.
9. Use equivalent representations of numbers such as fractions, decimals, and percents to facilitate estimation.
10. Determine whether a given estimate is an overestimate or an underestimate.

Building upon knowledge and skills gained in the preceding grades, and demonstrating continued progress in Indicator 6 above, by the end of **Grade 12**, students:

11. Estimate probabilities and predict outcomes from real-world data.
12. Recognize the limitations of estimation, assess the amount of error resulting from estimation, and determine whether the error is within acceptable tolerance limits.

STANDARD 11 All students will develop an understanding of patterns, relationships, and functions and will use them to represent and explain real-world phenomena.

Descriptive Statement

Patterns, relationships, and functions constitute a unifying theme of mathematics. From the earliest age, students should be encouraged to investigate the patterns that they find in numbers, shapes, and expressions, and, by doing so, to make mathematical discoveries. They should have opportunities to analyze, extend, and create a variety of patterns and to use pattern-based thinking to understand and represent mathematical and other real-world phenomena. These explorations present unlimited opportunities for problem-solving, making and verifying generalizations, and building mathematical understanding and confidence.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Reproduce, extend, create, and describe patterns and sequences using a variety of materials.
2. Use tables, rules, variables, open sentences, and graphs to describe patterns and other relationships.
3. Use concrete and pictorial models to explore the basic concept of a function.
4. Observe and explain how a change in one physical quantity can produce a corresponding change in another.
5. Observe and recognize examples of patterns, relationships, and functions in other disciplines and contexts.
6. Form and verify generalizations based on observations of patterns and relationships.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

7. Represent and describe mathematical relationships with tables, rules, simple equations, and graphs.
8. Understand and describe the relationships among various representations of patterns and functions.
9. Use patterns, relationships, and functions to model situations and to solve problems in mathematics and in other subject areas.
10. Analyze functional relationships to explain how a change in one quantity results in a change in another.
11. Understand and describe the general behavior of functions.

12. Use patterns, relationships, and linear functions to model situations in mathematics and in other areas.
13. Develop, analyze, and explain arithmetic sequences.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12,** students:

14. Analyze and describe how a change in an independent variable can produce a change in a dependent variable.
15. Use polynomial, rational, trigonometric, and exponential functions to model real-world phenomena.
16. Recognize that a variety of phenomena can be modeled by the same type of function.
17. Analyze and explain the general properties and behavior of functions, and use appropriate graphing technologies to represent them.
18. Analyze the effects of changes in parameters on the graphs of functions.
19. Understand the role of functions as a unifying concept in mathematics.

STANDARD 12	All students will develop an understanding of statistics and probability and will use them to describe sets of data, model situations, and support appropriate inferences and arguments.

Descriptive Statement

Probability and statistics are the mathematics used to understand chance and to collect, organize, describe, and analyze numerical data. From weather reports to sophisticated studies of genetics, from election results to product preference surveys, probability and statistical language and concepts are increasingly present in the media and in everyday conversations. Students need this mathematics to help them judge the correctness of an argument supported by seemingly persuasive data.

Cumulative Progress Indicators

By the end of **Grade 4,** students:

1. Formulate and solve problems that involve collecting, organizing, and analyzing data.
2. Generate and analyze data obtained using chance devices such as spinners and dice.
3. Make inferences and formulate hypotheses based on data.
4. Understand and informally use the concepts of range, mean, mode, and median.
5. Construct, read, and interpret displays of data such as pictographs, bar graphs, circle graphs, tables, and lists.

6. Determine the probability of a simple event, assuming equally likely outcomes.
7. Make predictions that are based on intuitive, experimental, and theoretical probabilities.
8. Use concepts of certainty, fairness, and chance to discuss the probability of actual events.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8**, students:

9. Generate, collect, organize, and analyze data and represent this data in tables, charts, and graphs.
10. Select and use appropriate graphical representations and measures of central tendency (mean, mode and median) for sets of data.
11. Make inferences and formulate and evaluate arguments based on data analysis and data displays.
12. Use lines of best fit to interpolate and predict from data.
13. Determine the probability of a compound event.
14. Model situations involving probability, such as genetics, using both simulations and theoretical models.
15. Use models of probability to predict events based on actual data.
16. Interpret probabilities as ratios and percents.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12**, students:

17. Estimate probabilities and predict outcomes from actual data.
18. Understand sampling and recognize its role in statistical claims.
19. Evaluate bias, accuracy, and reasonableness of data in real-world contexts.
20. Understand and apply measures of dispersion and correlation.
21. Design a statistical experiment to study a problem, conduct the experiment, and interpret and communicate the outcomes.
22. Make predictions using curve fitting and numerical procedures to interpolate and extrapolate from known data.
23. Use relative frequency and probability, as appropriate, to represent and solve problems involving uncertainty.
24. Use simulations to estimate probabilities.
25. Create and interpret discrete and continuous probability distributions, and understand their application to real-world situations.
26. Describe the normal curve in general terms, and use its properties to answer questions about sets of data that are assumed to be normally distributed.
27. Understand and use the law of large numbers (that experimental results tend to approach theoretical probabilities after a large number of trials).

STANDARD 13	All students will develop an understanding of algebraic concepts and processes and will use them to represent and analyze relationships among variable quantities and to solve problems.

Descriptive Statement

Algebra is a language used to express mathematical relationships. Students need to understand how quantities are related to one another, and how algebra can be used to concisely express and analyze those relationships. Modern technology provides tools for supplementing the traditional focus on algebraic techniques, such as solving equations, with a more visual perspective, with graphs of equations displayed on a screen. Students can then focus on understanding the relationship between the equation and the graph, and on what the graph represents in a real-life situation.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Understand and represent numerical situations using variables, expressions, and number sentences.
2. Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and number sentences, and translate from one to another.
3. Understand and use properties of operations and numbers.
4. Construct and solve open sentences (example: 3 + ___ = 7) that describe real-life situations.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

5. Understand and use variables, expressions, equations, and inequalities.
6. Represent situations and number patterns with concrete materials, tables, graphs, verbal rules, and standard algebraic notation.
7. Use graphing techniques on a number line to model both absolute value and arithmetic operations.
8. Analyze tables and graphs to identify properties and relationships.
9. Understand and use the rectangular coordinate system.
10. Solve simple linear equations using concrete, informal, and graphical methods, as well as appropriate paper-and-pencil techniques.
11. Explore linear equations through the use of calculators, computers, and other technology.
12. Investigate inequalities and nonlinear equations informally.
13. Draw freehand sketches of, and interpret, graphs which model real phenomena.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12**, students:

14. Model and solve problems that involve varying quantities using variables, expressions, equations, inequalities, absolute values, vectors, and matrices.
15. Use tables and graphs as tools to interpret expressions, equations, and inequalities.
16. Develop, explain, use, and analyze procedures for operating on algebraic expressions and matrices.
17. Solve equations and inequalities of varying degrees using graphing calculators and computers as well as appropriate paper-and-pencil techniques.
18. Understand the logic and purposes of algebraic procedures.
19. Interpret algebraic equations and inequalities geometrically, and describe geometric objects algebraically.

STANDARD 14	All students will apply the concepts and methods of discrete mathematics to model and explore a variety of practical situations.

Descriptive Statement

Discrete mathematics is the branch of mathematics that deals with arrangements of distinct objects. It includes a wide variety of topics and techniques that arise in everyday life, such as how to find the best route from one city to another, where the objects are cities arranged on a map. It also includes how to count the number of different combinations of toppings for pizzas, how best to schedule a list of tasks to be done, and how computers store and retrieve arrangements of information on a screen. Discrete mathematics is the mathematics used by decision-makers in our society, from workers in government to those in health care, transportation, and telecommunications. Its various applications help students see the relevance of mathematics in the real world.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Explore a variety of puzzles, games, and counting problems.
2. Use networks and tree diagrams to represent everyday situations.
3. Identify and investigate sequences and patterns found in nature, art, and music.
4. Investigate ways to represent and classify data according to attributes, such as shape or color, and relationships, and discuss the purpose and usefulness of such classification.
5. Follow, devise, and describe practical lists of instructions.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8**, students:

6. Use systematic listing, counting, and reasoning in a variety of different contexts.
7. Recognize common discrete mathematical models, explore their properties, and design them for specific situations.
8. Experiment with iterative and recursive processes, with the aid of calculators and computers.
9. Explore methods for storing, processing, and communicating information.
10. Devise, describe, and test algorithms for solving optimization and search problems.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12**, students:

11. Understand the basic principles of iteration, recursion, and mathematical induction.
12. Use basic principles to solve combinatorial and algorithmic problems.
13. Use discrete models to represent and solve problems.
14. Analyze iterative processes with the aid of calculators and computers.
15. Apply discrete methods to storing, processing, and communicating information.
16. Apply discrete methods to problems of voting, apportionment, and allocations, and use fundamental strategies of optimization to solve problems.

STANDARD 15 All students will develop an understanding of the conceptual building blocks of calculus and will use them to model and analyze natural phenomena.

Descriptive Statement

The conceptual building blocks of calculus are important for everyone to understand. How quantities such as world population change, how fast they change, and what will happen if they keep changing at the same rate are questions that can be discussed by elementary school students. Another important topic for all mathematics students is the concept of infinity — what happens as numbers get larger and larger and what happens as patterns are continued indefinitely. Early explorations in these areas can broaden students' interest in and understanding of an important area of applied mathematics.

Cumulative Progress Indicators

By the end of **Grade 4**, students:

1. Investigate and describe patterns that continue indefinitely.
2. Investigate and describe how certain quantities change over time.
3. Experiment with approximating length, area, and volume, using informal measurement instruments.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 8,** students:

4. Recognize and express the difference between linear and exponential growth.
5. Develop an understanding of infinite sequences that arise in natural situations.
6. Investigate, represent, and use non-terminating decimals.
7. Represent, analyze, and predict relations between quantities, especially quantities changing over time.
8. Approximate quantities with increasing degrees of accuracy.
9. Understand and use the concept of significant digits.
10. Develop informal ways of approximating the surface area and volume of familiar objects, and discuss whether the approximations make sense.
11. Express mathematically and explain the impact of the change of an object's linear dimensions on its surface area and volume.

Building upon knowledge and skills gained in the preceding grades, by the end of **Grade 12**, students:

12. Develop and use models based on sequences and series.
13. Develop and apply procedures for finding the sum of finite arithmetic series and of finite and infinite geometric series.
14. Develop an informal notion of limit.
15. Use linear, quadratic, trigonometric, and exponential models to explain growth and change in the natural world.
16. Recognize fundamental mathematical models (such as polynomial, exponential, and trigonometric functions) and apply basic translations, reflections, and dilations to their graphs.
17. Develop and explain the concept of the slope of a curve and use that concept to discuss the information contained in graphs.
18. Develop an understanding of the concept of continuity of a function.
19. Understand and apply approximation techniques to situations involving initial portions of infinite decimals and measurement.

STANDARD 16	All students will demonstrate high levels of mathematical thought through experiences which extend beyond traditional computation, algebra, and geometry.

Descriptive Statement

High expectations for all students form a critical part of the learning environment. The belief of teachers, administrators, and parents that a student can and will succeed in mathematics often makes it possible for that student to succeed. Beyond that, this standard calls for a commitment that all students will be continuously challenged and enabled to go as far mathematically as they can.

Cumulative Progress Indicators

By the end of **Grade 12**, students:

1. Study a core curriculum containing challenging ideas and tasks, rather than one limited to repetitive, low-level cognitive activities.
2. Work at rich, open-ended problems which require them to use mathematics in meaningful ways, and which provide them with exciting and interesting mathematical experiences.
3. Recognize mathematics as integral to the development of all cultures and civilizations, and in particular to that of our own society.
4. Understand the important role that mathematics plays in their own success, regardless of career.
5. Interact frequently with parents and other members of their communities, including men and women from a variety of cultural backgrounds, who use mathematics in their daily lives and occupations.
6. Receive services that help them understand the mathematical skills and concepts necessary to assure success in the core curriculum.
7. Receive equitable treatment without regard to gender, ethnicity, or predetermined expectations for success.
8. Learn mathematics in classes which reflect the diversity of the school's total student population.
9. Be provided with opportunities at all grade levels for further study of mathematics, especially including topics beyond traditional computation, algebra, and geometry.
10. Be challenged to maximize their mathematical achievements at all grade levels.
11. Experience a full program of meaningful mathematics so that they can pursue post-secondary education.

STANDARD 17 All students' mathematical learning will embody the concept that engagement in mathematics is essential, and that decision-making, risk-taking, cooperative work, perseverance, self-assessment, and self-confidence are frequently keys to success.

(This "learning environment standard" was developed and approved by the task force that prepared the *Mathematics Standards* and appears in the Introduction to the *Mathematics Standards* chapter of the New Jersey State Department of Education's *Core Curriculum Content Standards*; however, since it was not considered a "content standard," it was not presented to the New Jersey State Board of Education for adoption.)

Descriptive Statement

Engagement in mathematics should be expected of all students, and the learning environment should be one where students are actively involved in doing mathematics. Challenging problems should be posed and students should be expected to work on them individually and in groups, sometimes for extended periods of time, and sometimes on unfamiliar topics. They should be encouraged to develop traits and strategies — such as perseverance, cooperative work skills, decision-making, and risk-taking — which will be key to their success in mathematics.

Cumulative Progress Indicators

Experiences will be such that all students:

1. Demonstrate confidence as mathematical thinkers, believing that they can learn mathematics and can achieve high standards in mathematics, and accepting responsibility for their own learning of mathematics.
2. Recognize the power that comes from understanding and doing mathematics.
3. Develop and maintain a positive disposition to mathematics and to mathematical activity.
4. Participate actively in mathematical activity and discussion, freely exchanging ideas and problem-solving strategies with their classmates and teachers, and taking intellectual risks and defending positions without fear of being incorrect.
5. Work cooperatively with other students on mathematical activities, actively sharing, listening, and reflecting during group discussions, and giving and receiving constructive criticism.
6. Make conjectures, pose their own problems, and devise their own approaches to problem solving.
7. Assess their work to determine the effectiveness of their strategies, make decisions about alternate strategies to pursue, and persevere in developing and applying strategies for solving a problem in situations where the method and path to the solution are not at first apparent.
8. Assess their work to determine the correctness of their results, based on their own reasoning, rather than relying solely on external authorities.

STANDARD 18 All students will be evaluated using a diversity of assessment tools and strategies to provide multiple indicators of the quality of every student's mathematical learning and of overall program effectiveness.

(This "learning environment standard" was developed and approved by the task force that prepared the *Mathematics Standards* and appears in the Introduction to the *Mathematics Standards* chapter of the New Jersey State Department of Education's *Core Curriculum Content Standards*; however, since it was not considered a "content standard," it was not presented to the New Jersey State Board of Education for adoption.)

Descriptive Statement

A variety of assessment instruments should be used to enable the teacher to monitor students' progress in understanding mathematical concepts and in developing mathematical skills. Assessment of mathematical learning should not be confined to intermittent standardized tests. The learning environment should embody the perspective that the primary function of assessment is to improve learning.

Cumulative Progress Indicators

Experiences will be such that all students:

1. Are engaged in assessment activities that function primarily to improve learning.
2. Are engaged in assessment activities based upon rich, challenging problems from mathematics and other disciplines.
3. Are engaged in assessment activities that address the content described in all of New Jersey's *Mathematics Standards*.
4. Demonstrate competency through varied assessment methods including, but not limited to, individual and group tests, authentic performance tasks, portfolios, journals, interviews, seminars, and extended projects.
5. Engage in ongoing assessment of their work to determine the effectiveness of their strategies and the correctness of their results.
6. Understand and accept that the criteria used to evaluate their performance will be based on high expectations.
7. Recognize errors as part of the learning process and use them as opportunities for mathematical growth.
8. Select and use appropriate tools effectively during assessment activities.
9. Reflect upon and communicate their mathematical understanding, knowledge, and attitudes.

Nine Vignettes

This section contains nine vignettes which suggest how New Jersey's *Mathematics Standards* can be effectively implemented in classroom settings.

The table below indicates the content standards and grade levels which each vignette particularly addresses.

The vignettes highlight, using marginal notes, how the learning environment standards and the first five content standards serve as a context for mathematics learning. These reinforce the emphasis that the *why's* and *how's* of mathematics learning must be integrated with the content.

Although these nine vignettes reflect all eighteen standards, they certainly do not fully address all of the cumulative progress indicators that are attached to the standards. They are intended to be illustrations of the way that individual educators have suggested that these standards be implemented. Teachers are encouraged to review and discuss them, to experiment with practices that they exemplify, and to develop their own activities consistent with the standards.

		Content Standard															
Vignette	Page	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Grades K-4																	
Elevens Alive!	46	X	X	X	X	X			X				X				X
Product and Process	48	X		X	X	X	X		X		X						X
Sharing A Snack	51	X	X	X	X	X	X		X						X		X
Grades 5-8																	
The Powers of the Knight	54	X	X	X	X	X	X		X			X		X			X
Short-circuiting Trenton	56	X		X	X	X				X					X		X
Mathematics at Work	57	X	X	X	X	X		X		X	X						X
Grades 9-12																	
On The Boardwalk	60	X	X	X	X			X				X	X		X	X	X
A Sure Thing!?	64	X	X	X	X			X				X					X
Breaking The Mold	66	X	X	X	X	X		X		X	X			X		X	X

Elevens Alive!

The students:

While Mr. Johnson is meeting with some of the children in his first-grade class, others are involved in a number of different activities. At the Math Center, pairs of students have cups with eleven chips that are yellow on one side and red on the other. As each pair pours out the chips, they write a number sentence showing how many yellows and how many reds they got, as well as the total. When they have written ten number sentences each, they move on to another activity.

work on basic facts in the context of a problem and in relation to other areas of mathematics.

work in pairs with manipulatives.

Later in the day, as Mr. Johnson begins the math lesson, he asks the students to recall their discussion from the previous day, “What were we talking about yesterday in math?”

practice their number facts by writing down each result.

“We were doing numbers that add up to eleven, like *5+6* and *2+9*,” answers Clark.

share mathematical ideas.

“Or *3+8* and *4+7*,” adds Sarah.

“Is there more than one way to get a sum of eleven?”

Mr. Johnson lists all of the children's responses on the board. He goes on to ask them, “What were you doing at the Math Center earlier today?”

connect their understanding of one mathematical idea to another.

Jackie responds, “We were tossing counters and writing number sentences.”

“We were tossing eleven counters!” says Toni.

“What can you tell me about your results?” asks Mr. Johnson. “Did you get the same number sentences as your partner?”

“No — we got different ones!”

“Our answers were always the same — eleven!”

“I got some number sentences more than once!”

report and reflect on the differences of their results.

“I got 5+6 three times!”

“I didn't get 0+11 or 11+0 at all!”

“Why do you think you got different answers?” asks Mr. Johnson. He listens as the students talk about fairness, luck, and chance, pointing out that all of the counters are alike. The students agree finally that the different number sentences are a result of chance.

informally explore the concepts of probability.

The students continue their discussion of which number sentences appear more often than others. One of the children suggests that maybe they should make a graph to help them see which number sentences occur most often. Mr. Johnson thinks that this is a good idea. He goes through their list of number sentences, asking students to raise one finger if they got that number sentence once, two fingers if they got it twice, and so on. For each finger raised, he puts a tally mark on the board. When they are done, he asks whether there were any other number sentences that anyone got. Then the children look at the general shape of the data, noticing that most of the number sentences were in the middle. Mr. Johnson points out that not all of the number sentences are equally likely to occur. He says that tomorrow they will have a chance to play a game with the counters in which they will need to select which number sentences will be winners. Tomorrow's activity will continue providing opportunities for practicing basic facts while building on the beginning ideas of probability.

The students:

use different methods to display data.

make inferences about their data.

Product and Process

The students:

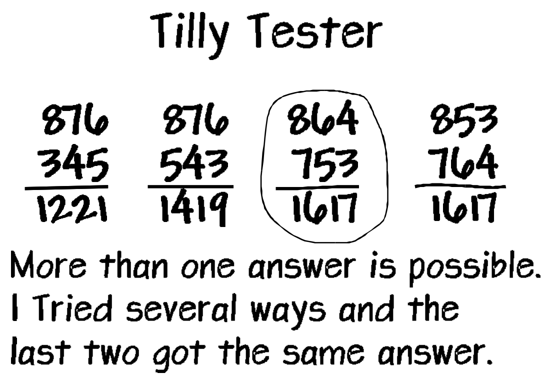
Mr. Marshall had assigned the following problem from the New Jersey Early Warning Test as a homework assignment for his fourth grade class:

> Use each of the digits 3, 4, 5, 6, 7, and 8 <u>once and only once</u> to form three-digit numbers that will give the largest possible sum when they are added. Show your work.
> Is more than one answer possible? Explain your answer.

are asked to respond to open-ended questions and present and defend their solutions.

The students were to solve the problem and match their response with that of Tilly Tester to see if they agree or disagree with Tilly's response and explain why.

are asked to analyze problems for reasonableness of results and to diagnose errors.



As the math class begins, Mr. Marshall allows the students to work in the cooperative learning groups which they have been working with this month to compare the results of their homework assignment. Mr. Marshall visits each group noting who has completed the assignment as well as the direction of the discussion for each group. Homework assignments are important and students are given credit for homework. Strategies such as displaying answers on the overhead projector and working in cooperative learning groups are used to ensure that homework review is no more than 5 to 10 minutes.

work cooperatively to assess their own and each other's work.

Mr. Marshall then asks the students to show the level of their agreement with Tilly's response on a 0-5 scale, with 0 signifying disagreement and 5 signifying total agreement. Most students raise 4 or 5 fingers, and the discussion then focuses on how Tilly's answer could be improved. One group notes that Tilly should have added each pair of numbers and shown the sum for each, while another group explains that Tilly could have also changed the hundreds place to get 754+863 and 763+854.

are willing to take a position without the fear of being incorrect.

At this point, Mr. Marshall discusses Tilly's understanding of place value and uses the opportunity to summarize the students' responses and lead into

use their knowledge of numeration to help solve problems.

the objective of the day which focuses on place value and multiplication.

"Let's work on multiplication today, and to get started, let's do some mental math with multiplication. On the back of your homework, number 1 to 10. Write the answers only for my mental math flashcards."

Individually, the students write answers for 8000x3, 6000x7+50, 300x7, etc. After the ten problems, Mr. Marshall has the students exchange papers, and they correct and discuss the answers. The papers are collected, and Mr. Marshall poses the following problem for his students:

> *Use four of these five digits and construct the multiplication problem that gives the greatest product: 1, 3, 5, 7, 9*

Before allowing the students to start work on the problem, he asks them to estimate what the largest product obtained in this manner might be. Students offer estimates ranging from 3000 to 10,000 and provide explanations for their guesses. When allowed to, the class works in their cooperative learning groups. Calculators are available, and some students start guessing and checking with their calculators.

One group begins to discuss which digits to use, wondering whether there would be a reason not to use the four largest digits. Another group is discussing whether a 2-by-2 or a 1-by-3 arrangement would be the best for getting a large product, an aspect of the problem that some groups have completely missed. Most of the groups get around to trying out sample problems of a variety of sorts to get some parameters worked out. Toward the end of the class session, the groups share the specific answers they have come up with. The three examples that are suggested are:

753	93	953
x 9	x 75	x 7
6777	6975	6671

It is clear to everyone that the 2-by-2 digit problem is the one with the greatest product, but Mr. Marshall is looking for some generalizations that can be made. He points out that none of the groups used the digit "1" in their examples. *Can the lowest digit always be ruled out?* He asks the groups that arrived at the 2-digit problem to explain how they decided where to put the individual digits. *Does it matter where they were placed? Where does the largest one go? The smallest? Do you think it would always work that way regardless of what the individual digits were? How can you check?* The students reflexively pick up their calculators and begin to formulate other versions of the problem that use other digits and to check which arrangements of the digits give the largest product. One student asks his partners what they think would happen if two or three

The students:

use mental math regularly throughout the curriculum.

demonstrate their understanding of mathematical concepts in a variety of ways, each of which provides valuable assessment information to the teacher.

are encouraged to estimate solutions before actually determining answers.

use calculators to aid in the problem-solving process.

using mathematical reasoning to formulate strategies and solutions.

approach numerical operations from a holistic point of view rather than only through paper-and-pencil manipulation.

digits were the same.

For homework, Mr. Marshall asks the students to use the same five digits, but to find the smallest possible product. They are then to write a paragraph describing their solution and the reasoning they used to show it is, indeed, the smallest product. Specifically, they are to consider the question: *Can you just turn your thinking about the way you got the largest product upside down and use it to get the smallest product?*

The students:

write paragraphs describing and justifying their positions.

Sharing a Snack

The students:

Today is November 12 and Maria, a student in Miss Palmer's second grade class is very excited. Today is Maria's birthday, and as is the custom in her class, she is bringing in a birthday snack to share with her classmates. Maria and her father spent much of the previous evening making a batch of chocolate chip cookies and she proudly walks into class carrying a cannister full to the brim. Miss Palmer realizes that she can use mathematics to help the class divide the cookies.

use mathematics to devise a solution for real-world problems.

Before the afternoon snack time, Miss Palmer poses the problem to the whole class.

Miss Palmer states, "Today is Maria's birthday and she has brought in some delicious chocolate chip cookies for all of us to enjoy at snack time. Maria told me she baked a whole bunch of cookies. I would like us to think about how we could determine the number of cookies each student in the class should get. Discuss it with your partner."

use cooperative work to generate potential solutions.

The students begin to discuss all of their ideas. After a few minutes, Miss Palmer calls on a few of the students. As they share their ideas, the teacher records them on the language experience chart.

regularly share their ideas publicly.

Sarah states, "Well, me and Mario think that the first thing we have to do is count the cookies to find out how many there are."

Jerome adds, "Yeah, and we also need to know how many children are in the class today."

"That's easy. I did the lunch count this morning and there are 22 children in school today," Maria volunteers.

Luis chimes in. "Once we know how many cookies and how many children, then we can figure out a way to solve the problem."

The children all agreed that since they know there are 22 children in class today, the next step was to determine the number of cookies. Miss Palmer highlights that idea on the language experience chart and gives each pair of children a bag of counters which represents the number of cookies.

Miss Palmer says, "Each pair of children received a bag of counters. I want you to pretend that these are Maria's cookies. I've counted the cookies. The number of counters in each bag is equal to the number of cookies Maria brought for a snack. With your partner, use your counters to first decide how many cookies Maria brought to school and then determine how many cookies each student will get if the cookies are to be shared equally among everyone in the class. When you are finished, each pair will need to write a

use manipulatives to model real-world situations.

story which explains how both of you solved the problem."

The children worked with their regular partners. The first task they all tackled was to count the number of counters in each bag. Most of the pairs of children counted by twos to determine the total number of counters was 62. However, Alex and Laura kept losing count when trying to count all the counters and decided to group the counters by ten. Miss Palmer was delighted to see that most pairs of children had written the total number of counters (62) on a sheet of paper. She had been stressing the importance of collecting data and recording information.

As Miss Palmer continued to circulate around the classroom, she noticed the children were solving the sharing problem in various ways.

One pair of students begins by drawing 22 stick figures to stand for the students in the class and then starts to "give out" the cookies by drawing them in their stick figures' hands. Another pair also starts with 22 stick figures but then draws 62 little cookies on another part of the paper and is stumped about where to go from there. Mario and Sarah begin to sort the 62 counters into 22 piles. Another pair, trying to use calculators to solve the problem, starts by adding 22 cookies for everyone to another 22 cookies for everyone to a third 22 cookies for everyone and then realizes that they have exceeded the number of cookies available.

Miss Palmer, noticing that the students will be unable to finish the problem before they have to go to Physical Education, calls the students back together.

"I want all of you to stop what you are doing, and with your partner write a story to tell me how you are attempting to solve this problem," she directs.

The students eagerly write their stories. Some use pictures to help illustrate their solutions.

Miss Palmer requests, "I would like some of the pairs to report to the whole class how they were attempting to solve the problem."

Luis states, "Well, Elizabeth and I figured out that each student could have 2 cookies and there will be 18 cookies left. We know this because we drew a picture of the class and put counters on each student. When we couldn't give counters to every kid, we decided those were leftovers and we counted them."

Lisa volunteers, "We drew stick figures too. After we gave out 2 cookies to each child, Jerome said we couldn't give out the 18 leftovers. But I think we can break the leftover cookies in half. Then each child would get 2 whole cookies and one half cookie. But I'm not sure how many would be left over then."

The students:

use their knowledge of decimal place value to simplify the task.

develop their own methods for solving the problem.

use technology as a problem-solving tool.

draw pictures to model their solutions.

give explanations of their strategies for solving the problem.

"Sarah and I used the calculator to solve the problem. We put in 62 and I counted while Sarah subtracted 22. We got 2 with 18 left over," Mario added.

"Alex and I got a different answer. We used the counters and put them into 22 piles, but we got 17 leftovers," Laura said.

Lisa suggested, "Maybe you and Alex should count them again to make sure you have the right number in each pile."

Laura and Alex recount their piles and discover that one counter fell on the floor.

Vanessa states, "Me and my partner thought of another way of sharing the leftover cookies. Everyone could write their name on a piece of paper, then put all the papers into a bag and have Maria close her eyes and pick out 18 names. Those kids would get the extra cookies."

Sarah protested, "We forgot about Miss Palmer. We should give her 2 cookies and that would leave 16 left over. Maria could give them to the principal and the other ladies in the office."

Miss Palmer wrapped up the discussion. "We've discussed many ideas for sharing the 62 cookies Maria brought for a snack. On the back of the sheet of paper I gave you, I would like you and your partner to decide on how you think we could fairly share the cookies."

The children work on their final summary of the problem and hand their papers in before getting on line for Physical Education. While the children are in Physical Education, Miss Palmer reads the children's solutions. She makes notes on the cards she keeps for each child. This will help her better understand various developmental levels of her students. She notices that Vanessa has really made progress since September. Laura and Alex still like to "rush" to finish their work. She makes a note on their paper encouraging them not to be so concerned about being the first ones finished. Overall, she feels encouraged, not only about the solutions to the problems, but also about the ways in which her class has learned to communicate their ideas both orally and on paper. She decides to let the class choose one of the methods suggested to distribute the cookies at snack time.

The students:

informally explore the uses of fractions and notions of fair sharing.

are mutually supportive and regularly offer feedback to each other.

demonstrate their understanding of mathematical concepts in a variety of ways, each of which provides valuable assessment information to the teacher.

The Powers of the Knight

The students:

Mr. Santos' 6th grade class has just completed a review of place value in the decimal number system and he is preparing to start a unit introducing exponents. He has coordinated the timing of this unit with the language arts teacher whose class is in the midst of a unit on fables. One fable they have read involves a knight who saves a kingdom from a horrendous dragon. Given the opportunity to determine his own reward, he tells the king that he would take one penny on the first square of a chessboard, two pennies on the second, four pennies on the third, and so forth until each square on the chessboard has twice as many as the previous one. Mr. Santos has the students recall the story and then asks the students to determine how much money the knight would make with this method of payment.

are connecting a language arts experience to their mathematics learning.

Mary said, “We need to know how many squares there are on a chessboard before we can do this problem.”

are comfortable taking risks.

Lionel stated, “Give me a minute to think. I play on the chess team, but I need to take a moment to picture it. Let's see, I know it's square and there are 1,2,3, ...8 squares along the one side. There are 64 squares!”

use known facts to explain their thinking.

Jerry shouted, “He gets 128 pennies. Two on each square.”

“The fable doesn't say he gets two on each square! It says that each square has twice as many as the one before. It has to be more than 128!” corrected Meredith.

react substantively to others' comments.

“We need to examine this situation in some organized fashion. I want you to get in your groups of four and determine the people who will serve the usual roles of leader, recorder, reporter, and analyzer of group interaction,” stated Mr. Santos.

use standard cooperative learning strategies.

One group decided to develop a computer program which printed a table listing the number of the square, the number of coins on that square, and a subtotal to that point.

use technology to help solve the problem.

Another group borrowed the class chessboard and began placing play coins on the squares. It soon became obvious to them that they would not have enough play money to complete this attempt. They started to make a table with the information they had constructed and worked to find a pattern which they could extend to the complete board. Their table only included columns showing the number of the square, the number of coins on that square, and a column to list patterns. They discovered that the number of coins could be represented by raising 2 to the power which was one less than the number of the square. Using calculators, they found the number of coins on each square and then the total number of coins.

concretely model the problem before they move on to more symbolic procedures.

use self-assessment to determine the effectiveness of their method.

Another group began making a table similar to the group above, but they also included a column showing the partial sums and another which attempted to find a pattern in the partial sums. Eventually, they discovered that the partial sum at each square was one less than 2 raised to the power equal to the number of the square. They could then quickly utilize the calculators to compute the total.

At the end of the period, Mr. Santos reminded the groups that they were to prepare a report of their methods which included a description of their processes, an explanation of why they chose them, and their evaluation of their processes. He asked each of them to consider the magnitude of their answer and find some way to explain to another person just how large the answer was. Students brainstormed some ideas such as the distance between two known points or objects, the magnitude of the national debt, and the number of people on earth.

The students:

analyze mathematical situations by recognizing and using patterns and relationships.

choose technology to reduce the computational load.

write about their approaches and solutions to problems.

connect their knowledge of mathematics to the real world.

Short-circuiting Trenton

The students:

Ms. Ramirez announces to her seventh grade class that in three weeks they will make a journey to Trenton, the capital of New Jersey. They will be visiting eight sites — the Capitol, the New Jersey Museum, the War Memorial, the Old Graveyard, Trent House, the Old Barracks Museum, the Firehouse, and the Pedestrian Mall. To ensure that they spend as much time at the sites as possible, and do as little walking as possible, the class must find the most efficient walking tour for the trip, starting and ending at the parking lot.

apply mathematical skills to solve a real-world problem.

The first problem that the students must address is finding the walking distance between each pair of sites. Ms. Ramirez supplies each team with a street map and a ruler; the maps identify all the sites to be visited and the routes joining them. She assigns each group the task of finding the distances between one site and all the others. This turns out to be an interesting task, since different groups interpret it differently. Some groups, for example, measure the straight line distance between two sites forgetting that buildings or ponds might render that walk impossible. How to measure the walking distance thus becomes an important topic of discussion, as does the question of appropriate units. These questions are eventually settled and the teacher uses the students' measurements to write a matrix which indicates the walking distance between any two of the eight sites; different groups occassionally have obtained different numbers, but after discussion, they have arrived at a common answer.

use cooperative group work to generate problem-solving strategies.

freely exchange ideas and participate in discussions requiring higher-order thinking.

collect and organize data needed to solve the problem.

Ms. Ramirez selects a sample route for the walking tour and through discussion with the class explains how the total length of the walking tour is obtained from the matrix of information that the students generated — you find the distances between consecutive sites on the tour, and then add up the walking distances along the tour. She now asks her students to work in groups to decide on a strategy that they think will produce an efficient route (which starts and ends at the parking lot), and to assist the group's recorder in writing a short paragraph explaining their strategy. Some groups decide to list all possible routes and calculate how long a walk each route entails. (Ms. Ramirez asks the students how many possible routes do they think they will have to list.) Other groups suggest that the best route is obtained by always going to the nearest site.

recognize there are numerous ways to solve the problem.

work in cooperative groups to develop alternative strategies.

Ms. Ramirez now asks the students to use calculators to carry out their strategy and determine the travel time for the routes they will be considering. After each group presents its results, the class will together compare the various methods that were proposed and the accompanying results. Among the questions which Ms. Ramirez will ask are: "Do the various methods give the same result?", "Which methods result in a most efficient route?", "What other strategies could we have used?" Responses from the students might include: "always use the shortest distance", "never use the longest distance", "put distances in increasing order and use only those that neither make a loop or put a third edge into a vertex."

compare the variety of strategies proposed.

Mathematics at Work

The students:

As a regular feature in his class, Mr. Arbeiter has parents of each student make a presentation about their job and how the various educational disciplines are needed for them to be successful. Today, Emily has asked her Mom, the owner of a heating and air conditioning company, to talk to her class. Mrs. Flinn and Mr. Arbeiter decide to have the students help her solve a problem similar to one which her company faces regularly. She briefly describes her company, the work that she does, and tells the students that they are going to help her determine how large an air conditioner will be needed in the classroom. She poses the following problem: What information about the room would be most important in determining how large an air conditioner is needed? The students quickly agree that the amount of air conditioning would depend on the amount of air in the room, and that in turn, depended on how much space there was in the room. Through suggestions and hints, Mrs. Flinn had them realize that the amount of sunlight entering the room would have an effect as well and they quickly agreed that the area of the windows must be found too.

interact with parents who use mathematics and other disciplines in their daily lives.

have the time to explore a problem situation thoroughly.

Mr. Arbeiter reminded the class that there is a mathematical term which represents the amount of space, and asked each student to write down that term. As was his custom, Mr. Arbeiter asked six students, one quarter of the class, to read the words they had each written. Four read the word "volume" and two read "area." By a show of hands, he found that about one third of the class had written "area" and two thirds had written "volume." In their groups, the students were asked to discuss the difference between area and volume and to write down the differences between them. As the groups discussed these concepts Mrs. Flinn and Mr. Arbeiter circulated among them, making sure that each group had focused on the difference between area and volume; subsequently the groups read the statements they had prepared, and the entire class discussed and commented on the groups' statements. Mrs. Flinn had the class discuss which of the concepts were needed on the two phases; amount of space in the classroom and how much window space there was.

are regularly assessed through a variety of methods.

work in a variety of settings to develop concepts and understanding.

Now that all students agreed on the difference between area and volume and where each applied in this case, the discussion turned to discussion centered on how one obtains the volume of the classroom and the area of the windows. Although familiar with the concept of volume, the class was not able to calculate volume easily, so Mr. Arbeiter suggested that each group build a rectanguar box out of cubes and figure out how many cubes the box contained. Most groups discovered that they could get the answer by multiplying the number of cubes in the bottom layer by the number of layers (the "height"), and agreed with Mr. Arbeiter's conclusion that V = BxH. When Mr. Arbeiter asked them how they calculate the number of cubes in the bottom layer, all agreed that you multiplied length times width; and when the teacher wrote V=BxH=(LxW)xH, several other groups recognized that that was how they found the volume of their box.

use concrete materials to develop a model for volume.

Mr. Arbeiter asked the class "How does the volume formula help us find the volume of the classroom?" The students agreed that the shape of the classroom was about the shape of a rectangular box, but were quick to point out that to any answer obtained by the formula would have to be considered an estimate, since it would not be taking alcoves and pillars into consideration. They agreed to change the question to "How does this formula help us estimate the volume of the classroom?"

The students:

"All we have to do is measure the three quantities — length, width, and height, the three dimensions of the classroom, and multiply the three numbers together" was the prevailing sentiment. Marcia observed that "since we're only going to get an estimate anyway, why should we measure those three amounts exactly?" And Mrs. Flinn noted that her sales people often estimated the size of the room without making any measurements. "How can we estimate the dimensions of a room without making measurements?", she asked. Paula suggested that "maybe the salesperson estimates the three dimensions and multiplies those estimates together." "A great suggestion," Mrs. Flinn responded. "Let's try that ourselves."

recognize and apply estimating to geometric situations.

"Let's first estimate the *width* of the room. About how many inches wide is this room?" Brian pointed out that inches is an appropriate unit for a piece of paper, but not for a room. After a brief discussion, Mrs. Flinn revised her question to "About how many *feet* wide is this room?"

are exposed to a variety of open-ended questions and respond .

The students wrote down their estimates and explanations of how they arrived at them. After hearing all of the students estimates and reasons, the students were asked to return to their regular groups and decide as groups what they thought the width of the room was. "Well," said Mr. Arbeiter, "you all gave good reasons for your estimates, but now let's see whose estimate was closest. We'll measure the width of the classroom." Great cheers were heard for the groups whose estimate was closest to the actual measurement. The same process was repeated for length, and width as well as estimating the window area of the classroom. Mrs. Flinn pointed out that estimates were getting closer to the actual measurements each time they did it. She then showed the class a formula used to determine the number of BTUs needed for a room in terms of the volume of the room and the area of the windows. The data obtained by the class for the volume and window area was entered in the formula, and a quick calculation gave the number of BTUs needed for the classroom. Mrs. Flinn wrapped up her presentation by making the connection between the size needed, the cost of the purchase, and the regular expense of running the air conditioner. She emphasized that the success of her business rested on the sales people and their ability to estimate the needs well.

feel comfortable identifying errors.

communicate their answers and defend their thought processes.

examine the correctness of their results.

Mr. Arbeiter thanked Mrs. Flinn for her presentation and asked the students how they would like to practice the skills they had discussed today. Feeling confident, the students volunteered to estimate the data for their other classrooms. Mr. Arbeiter agreed to display the results, so long as the students agreed to leave off estimating while their other classes were in session.

The students:

extend their skills through practice in similar problems.

On the Boardwalk

The students:

"It isn't fair!", Jasmine announced to her class one Monday morning. "I used up $10 worth of quarters playing a boardwalk game over the weekend at the shore, and I only won once. And all I got for winning was a lousy stuffed animal!"

Ms. Buffon often told her class that mathematics was all around them, and had encouraged them to see the world with the eyes of a mathematician. So she wasn't surprised that Jasmine shared this incident with the class.

recognize the role that mathematics can play in explaining and describing the world around them.

"Please explain why you thought there was mathematics here," Ms. Buffon asked Jasmine.

"Well, first of all, I threw the quarters onto a platform which was covered with squares, you know, like a tile floor, so that reminded me of geometry. And as I was throwing my quarters away, one after another, I was reminded of all the probability experiments that we did last year, you know, throwing coins and dice. It wasn't exactly the same, but it was like the same."

connect previously learned mathematics to the current situation.

"Those were very good observations, Jasmine," said Ms. Buffon, "you recognized that the situation involved both geometry and probability, but you didn't tell the class what you had to do to win the game."

"Oh, you just had to throw the quarter so that it didn't touch any of the lines!" Ms. Buffon asked Jasmine to go to the board to draw a picture, explaining to her that not everyone will visualize easily the game she was talking about.

use different forms of communication to define a problem and share their insights.

Every other Monday, Ms. Buffon began her geometry class with a sharing session. Sometimes the "mathematics situations" that the students shared did not lead to extended discussions, in which case Ms. Buffon continued with the lesson she had prepared. But she was prepared to use the entire period for the discussion, and even carry it over into subsequent days, if the students got interested in the topic.

are afforded the opportunity to fully explore and resolve mathematical problems.

"Why didn't you think the game was fair?" she asked Jasmine. Jasmine repeated what she had said earlier, that she should have won more often and that the prizes should have been better. Other students in the class were asked to respond to the question, and after a lively interchange, they decided that for the game to be really fair, you should get about $10 in prizes if you play $10 in quarters; but, considering that they were having fun playing the game, and considering that the people running the game should get a profit, they would be satisfied with about $5 in prizes. Jasmine listened to the conversation intently, and chimed in at the end "That lousy bear wasn't worth more than a dollar or two!"

explore questions of fairness, geometry, and probability.

Moving the discussion in another direction, Ms. Buffon said "Now that we understand that it is possible to explain 'fairness' mathematically, let us investigate Jasmine's game to see if it really was unfair. What do you think were Jasmine's chances of winning a prize?"

The students:

This question evoked many responses from the class, and after some discussion the class agreed with Rob's comment that it all depended on the size of the squares. Jasmine did not know the actual size of the squares, so the class agreed that they might as well try to figure out the answer for different size squares. Dalia pointed out that this looked like another example of a function, and Ms. Buffon commended her for making this connection to other topics they had been discussing.

are encouraged to make connections to other topics within mathematics.

Returning to her previous question, Ms. Buffon suggested that the students do some experiments at home to help determine the probability of winning a prize. Each pair of students was asked to draw a grid on poster board, throw a quarter onto the poster board 100 times, and record the number of times the quarter was entirely within the lines; to simplify the problem, quarters that landed off the grid were not counted at all. Different students chose different size grids, ranging from 1.5" to 3.5", at quarter inch intervals.

model problems and conduct experiments to help them solve problems.

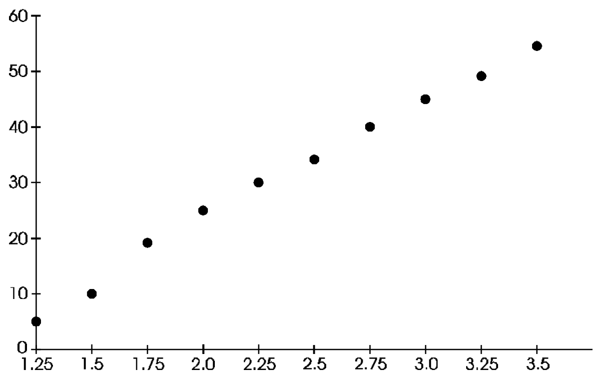
After school, Ms. Buffon visited the Math Lab where she spent some time trying to find materials related to this problem. When she looked under "probability" in the indexes of various mathematics education journals, she was led to several articles discussing geometric probability, which she learned is a branch of mathematics which addresses problems like Jasmine's game. With these resources available to her, Ms. Buffon no longer feels that she has to have all the answers, and can entertain discussions about mathematical topics with which she is unfamiliar. Tomorrow she will be able to tell the class what she has learned!

The next day the students reported on their results, and Ms. Buffon tabulated them in the following chart, and, at the same time, plotted their results on a graph:

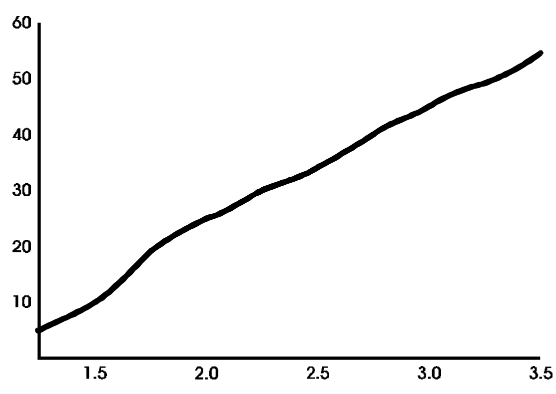
Size of Squares	Number of Wins
1.25	5
1.5	10
1.75	18
2	25
2.25	30
2.5	34
2.75	40
3	45
3.25	48
3.5	54

collect, analyze, and make inferences from data.

"Do you see any patterns here?", Ms. Buffon asked the class. They all agreed that, as expected, the larger the size of the squares, the more frequently Jasmine would have won the game. "What do you think the size of the squares were on the boardwalk game?" Ms. Buffon asked next. Everyone agreed that the squares were most likely smaller than



1.25" since her one prize out of forty quarters corresponded to 2.5 wins out of a 100 games, which was lower than obtained for the smallest squares in the experiment. Turning to the graph, Ms. Buffon asked “What would have happened if we tried the experiment with squares smaller than 1.25?" The students laughed, one after another, as they realized that if the size of the squares were small enough, you would never win the game. “Well, then, what would have happened if we tried the experiment with larger and larger squares?” Looking at the graph, the class found this a difficult question, but Fran broke the group's mindset by saying “Yeah, suppose the squares were as big as this room?” Then everyone realized that if the squares were larger and larger, you would become almost certain to win the game. “Dalia, do you remember your comment yesterday, that it sounded like we were working on a function?” Ms. Buffon asked. “Would you sketch the graph of that function for the class?” Ms. Buffon made a mental note to discuss this problem with her precalculus students, since she had many questions to ask them about this graph.



“Well, we've gotten a lot of information by using experimental methods about

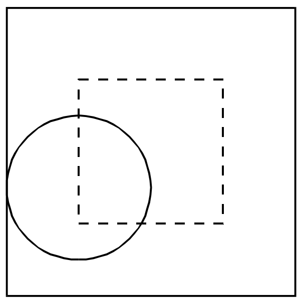
The students:

recognize connections between numerical patterns and functions.

Jasmine's game; let's see if we can figure out the probability theory behind it as well." At this point, Ms. Buffon was eager to tell her class what she had learned at the Math Lab. However, being aware that the students will grasp the solution method better if they have an opportunity to discover it for themselves, she asked the class to discuss the following question in their study groups: "How can you tell from the position of the quarter whether or not you would win the game?" Going from group to group, Ms. Buffon listens to the discussions. When the groups have discovered that to win the game, the center of the quarter must be sufficiently far from the closest border, she gives the groups their next task — to describe where in the square the center of the quarter must be.

The students:

formulate and test mathematical conjectures.



construct a pictorial model to represent the problem.

By reasoning in different ways, the groups all arrived at the same picture involving a smaller square inside the original square, and at the same conclusion — that you win if the center of the quarter lies inside the smaller square. With this information, the students are able to calculate the probability for any particular size of the square, and even to write an equation for the function whose graph they sketched earlier.

Having found the probability of winning the game, Ms. Buffon planned to return to the question that began this whole discussion — whether Jasmine's game was fair. But that was the topic for another day.

Note: Ms. Buffon realized that the graph of the function was not linear, as depicted earlier, even though the data seemed to indicate linear growth. With her precalculus students, she would have them translate the above situation into the equation $y=(x-d)^2/x^2$, where d is the diameter of the quarter. Then she would have them graph the function, enabling them to discover that although the graph appears to be linear, in reality it increases at a decreasing pace, and goes asymptotically to the line y = 1.

A Sure Thing!?

Ms. Jackson is teaching her geometry students to use and identify inductive reasoning.

She asks each student to draw a large triangle on their paper. She then asks the students to hold up their triangles so that they can see the wide variety that have been created. The students observe that all the triangles are different.

Ms. Jackson then asks the students to cut out their triangle, to tear off the corners of their triangle and to place the corners together so that they are adjacent. She circulates around the room to be sure everyone is on task, and tells students to record a description of what they see in their notebooks.

Ms. Jackson then asks a representative sampling of students to tell the class what they observed after fitting the corners together. The students report that it looks as if the corners form a straight line. Everyone agrees.

Ms. Jackson now asks the students to write a generalization about the angles of ANY triangle based upon the class results of this activity. She asks another representative sampling of students to state their generalizations. The students conclude that the sum of the measures of the angles of ANY triangle is 180 degrees.

She gives the students a definition of inductive reasoning. They recognize that they have used induction to reach their generalization about the angles of a triangle. She then asks them to think about when they have used inductive reasoning in the past and write an example in their notebooks.

Volunteers are asked to share their recollections with the rest of the class. Some are funny and some quite poignant. The teacher asks if anyone can see a drawback to inductive reasoning within social as well as mathematical contexts.

The class decides that one drawback is that you can't check all examples - all triangles cannot be checked to see if the angles always add to 180 degrees. Another is that if you check too few examples you might reach an erroneous conclusion. They discuss how this is the reason for much of the racial and gender stereotyping that they encounter. Ms. Jackson asks students to identify counterexamples for racial and gender stereotypes.

Ms. Jackson then asks the students to do another experiment. They use their compasses to draw 5 circles. On the first circle, the students identify and connect 2 points with a chord. They then state the number of non-overlapping regions into which the circle has been divided. On the second circle, students identify three points and draw all chords connecting these points. Once again, they state the number of non-overlapping regions into

The students:

use a variety of types of mathematical reasoning to solve problems.

are encouraged to form generalizations based on observations they have made.

are regularly asked to write about their understandings of mathematics and its uses in the real world.

which the circle has been divided. They continue this procedure until they find the number of non-overlapping regions formed when 5 points on the circle are fully connected by chords. Students record their data in a table and use inductive reasoning to predict the number of non-overlapping regions produced by fully connecting n points on the circle with chords:

The students:

# of Points	# of non-overlapping regions
2	2
3	4
4	8
5	16
n	$2^{(n-1)}$?????

generate a set of data and use pattern-based thinking to formulate solutions.

They are asked to state their conclusion in narrative form.

The students agree that the number of non-overlapping regions produced by fully connecting n points on a circle with chords is $2^{(n-1)}$. Students then test their conclusion by carrying out the experiment with 6 points. Many find their conclusion is wrong for n=6. They fully expected to find 32 regions but only got 31!

validate conclusions by looking for counterexamples.

As class draws to a close, Ms. Jackson gives a homework assignment in which students will induce as well as produce counterexamples to conclusions. Students leave class somewhat dazed by the last experiment. Many of them tell Ms. Jackson that something must be wrong because they are sure the answer is 32. They tell her that they will prove her wrong by reenacting the experiment at home. She looks delighted and encourages their pursuit.

Breaking the Mold

The students:

Mr. Miller wants his ninth grade mathematics class to review the rectangular coordinate system, reinforce how mathematics is used to model situations, and develop the concept of exponential functions. He decides this would be an excellent opportunity to utilize a real-world situation. He elects to build his effort around an experiment involving mold growth found in an old SMSG book entitled *Mathematics and Living Things.*

incorporate scientific applications in their study of mathematics.

At the beginning of the unit, Mr. Miller presents the class with a packet of required readings, each of which deals with growth patterns of living things. There is an article on the rabbit population of Australia, another on world population, and another on the spread of AIDS. He explains the goals of the unit, gives the expectations for the readings, and describes the purpose of the experiment the class will conduct. Mr. Miller has students distribute the lab directions and materials, and he has them prepare the medium for the mold growth.

LAB DIRECTIONS

Materials:

1 - 9-inch circular aluminum pie plate
2 - sheets of 10x10-squares-to-the-inch graph paper
1 - rubber band
glue
scissors, ruler
saran wrap
mixture of clear gelatin, bouillon, and water

Directions:

Cut one piece of graph paper to fit the bottom of the tin as closely as possible. Draw a set of axes with the origin as near the center as possible. Cement the paper to the bottom of the tin with rubber cement. Pour the mixture into the tin so as to cover the graph paper with a thin layer. Allow the tin to sit 5 minutes, cover with plastic wrap, and hold in place by a rubber band. Place the tin in a dark place where the temperature is fairly uniform.

On each day over the next two weeks, students record an estimate of the area covered by the mold, the increase in the area from the previous day, and the percent of increase. On Fridays, they are asked to extrapolate the growth they expect to occur on Saturday and Sunday and then interpolate the same information from the growth they see on Monday. They are required to maintain a graph of the percent of increase versus the days. The extrapolated and interpolated points are both graphed with special marks such as “X” or “O.”

estimate area of irregular figures.

collect and analyze data.

During the period of data-gathering, Mr. Miller develops exponential growth through the concept of compound interest and uses a graphing calculator to illustrate the graph of such growth. Each student is asked to suggest a function which would yield something close to their data, and has the opportunity to put their function into the graphing calculator and revise it until they are satisfied with the estimate. Time is provided to have the students discuss their reactions to the readings.

At the end of the two-week period, Mr. Miller has the students prepare a report relating the graph of their observations to the discussions of the readings and the work on compound interest. To extend the ideas developed in this experiment, students are given different data sets which came from actual measurements of various types of growth. Students work in groups, each group taking one of the sets of data. The groups are expected to make a presentation discussing the exponential function which models the growth, what limiting factors could be involved, and the carrying capacity of the environment.

As a closing activity, students are asked to choose a country from around the world, examine population growth over some period of time, and write a paper for inclusion in their portfolio discussing the mathematical issues and biological issues involved as well as a general discussion of the impact of such growth on the history of that period.

The students:

use technology as a tool of learning.

spend the time needed for mathematical discovery.

write about their understandings of the connections between mathematics and physical phenomena.

extend their understanding of mathematical concepts through cooperative work and presentation.

are assessed through alternative means.

explore the uses of mathematics in other disciplines.