Questions raised by Quora readers, related to set theory, each followed by my response:

How do you show that the set of cardinals less than a certain cardinal is uncountable (elementary set theory, set theories, ordinals, math)?

The claim is false in general. For example, the set of cardinals less than aleph-zero, the first infinite cardinal consists of the natural numbers N (including 0), which is countable.

What about the set of cardinals less than aleph-one, the first uncountable cardinal? That consists of N and aleph-zero. Hmm. That's still countable. And so it goes. The set of cardinals less than aleph-n is countable for each natural number n, even though the sequence of cardinals aleph-n get larger and larger.

The cardinal numbers are indexed by the ordinal numbers, and you have to go all the way out to aleph-sub-omega-sub-one, where omega-sub-one is the first uncountable ordinal number, before you find a cardinal number for which the set of smaller cardinals is uncountable.

There are a lot of uncountable cardinal numbers which have only a countable number of predecessors.

Surprise! That was not the expected outcome.

Is {1} a subset of {1, {2}}?

I'll answer your question with a second question: Is {2} a subset of {1, {2}}? If you contemplate these two questions, and recognize how they are the same and how they are different, you will be able to answer both of them.

Is Georg Cantor's continuum hypothesis considered completely solved or is there still no consensus on whether or not Cohen's work is a sufficient proof? If it is the latter, why is this so and is there anything that can be done to complete the proof?

For Cantor, the continuum hypothesis was about the set R of real numbers that we know and love. For him, the continuum hypothesis was the statement that every infinite set A of real numbers was either countable, and so could be placed in a 1–1 correspondence with the natural numbers N, or was the same size as R, and so could be placed in a 1–1 correspondence with R. Many mathematicians tried to prove or disprove the continuum hypothesis. For example, Sierpinski constructed the middle-third set (which later morphed into a main antecedent of the notion of fractals) as an attempt to find a set A of real numbers that was between N and R. Sierpinski's attempts to refute the continuum hypothesis failed, as did the attempts to prove it. So we still don't know whether Cantor's continuum hypothesis is true or false.

For Godel and Cohen, the continuum hypothesis was not about the real numbers R that we know and love, but about the axioms of set theory. Godel showed that if you take a reasonable set of axioms for set theory, then there is a model of those axioms in which the continuum hypothesis is true — that is, that the continuum hypothesis is consistent with the axioms. Cohen showed that there is a model of those axioms in which the continuum hypothesis is false — that is, that the continuum hypothesis is independent of the axioms. In neither Godel's model nor Cohen's model is the real numbers the same as the set R of real numbers that we know and love. (Some might claim that R is actually a figment of our imagination.)

Where we stand as a result of Godel's and Cohen's work is that we don't have a reasonable set of axioms for set theory that will resolve Cantor's continuum hypothesis. According to Godel's theorem, we could consistently add Cantor's continuum hypothesis to the axioms, and then we could prove Cantor's continuum hypothesis to be true. Or, according to Cohen's theorem, we could consistently add the negation of Cantor's continuum hypothesis to the axioms, and then conclude that Cantor's continuum hypothesis is false. But neither of those options would resolve the hypothesis that Cantor expressed, and evidently thought was correct. Moreover, axioms are usually thought of as self-evident statements and neither the continuum hypothesis nor its negation is self-evident.

The results of Godel and Cohen were theorems about axiomatic set theory, proving that those axioms failed to resolve an important question, and not really theorems about Cantor's continuum hypothesis.

P.S. I was present when Paul Cohen publicly announced the independence of the continuum hypothesis at Berkeley in 1963. This appropriately took place on the 4th of July, which is Independence Day in the United States.

If the null set is a subset of itself, and is that subset equal to the null set and thus have a subset of itself?

Your first question was "Does the null set contain itself?"

As has been noted in previous responses, the word "contains" is used in two ways — "contains as an element" and "contains as a subset." From the context you have to determine in which way it is being used.

The null set does contain itself as a subset, for any element of the null set (there are none) is an element of the null set. In the same way, every set A contains itself as a subset, since every element of A is an element of A.

On the other hand, the null set does not contain itself as an element, since the null set has no elements. So in this sense of contains, the null set does not contain itself.

Now, on to the present question. We have said that the null set contains itself as a subset. That subset is equal to the null set. The question seems to be confusing the relation “is a subset of” with the relation “is a proper subset of”.

We say that A is a subset of B if every element of A is an element of B. Thus every set A is a subset of itself. The null set is a subset of itself.

We say that A is a proper subset of B if A is a subset of B and is not equal to B. Thus no set A is a proper subset of itself. The null set is not a proper subset of itself.

In fact, the null set has no proper subsets because it is its only subset. The null set is unique in this respect. It is the only set that has no proper subset of itself. For if A is not the null set, then the null set is a proper subset of A.

How exactly does the axiom of comprehension in set theory resolve Russell’s paradox? It seems like it just says that we can only form “new” sets from subsets of pre-existing sets, but what stops those pre-existing sets from having the same problems?

Russell’s Paradox results from forming a set B all of whose elements are not elements of themselves, that is, B = { x | x not element of x}. We ask if B is an element of B. If B is an element of B, then, by the definition of B, B is not an element of itself, that is, B is not an element of B. That’s impossible because B is assumed to be an element of B. So B is not an element of B. But then, since B is an element of itself, it must be an element of B. That’s impossible too. In either case, whether B is an element of B or B is not an element of B, we reach a contradiction. That’s Russell’s Paradox.

One reason for the paradox is that we were allowed to create a set of all things that had a certain property — in the scenario above, { x | x is not an element of x}.

Let us insist that any set constructed in this way must be a subset of an existing set A. That is, we can form B = { x in A | x is not an element of x}, but not B = { x | x is not an element of x}.

As before, we ask if B is an element of B. If B is an element of B, then, by the definition of B, B is not an element of itself, that is, B is not an element of B. That seems again to lead to Russell’s Paradox.

But wait! There are two requirements for an element to get into B. One is that it is not a member of itself, the other is that it must be an element of A. So the conclusion is not that B is not an element of itself — that leads to a contradiction — but rather that B is not an element of A — and the contradiction is avoided.

Charles C. Pinter's book on set theory says that almost every set is a set of sets. Does that mean that numbers themselves are sets? If so, how?

One of the exquisite characteristics of set theory is that one can define all mathematical objects in terms of sets. For example, we can define the ordered pair (a,b) as the set {a, {a,b}} — the set with two elements one of which is the set whose only element is a, and the other of which is the set whose only two elements are a and b. With this definition, it is easy to show that two ordered pairs (a,b) and (c.d) are equal if and only if a=c and b=d.

Then one can define a relation from A to B as a set of ordered pairs each of whose first elements are in A and whose second elements are in B. ("First element" is well-defined since (a,b) and (b,a) are only equal when a=b.)

Then one can define a function from A to B as a relation from A to B which has certain specified properties.

Thus any function is a relation which is a set of ordered pairs each of which is a set with two elements.

So you can see why this textbook says that almost every set is a set of sets.

One can even define whole numbers to be sets, although not every treatment of set theory takes this step. Thus one can define 0 to be the empty set, usually denoted by phi, and one can inductively define each whole number greater than 1 to be the set of all preceding whole numbers. Thus 1 could be defined to be {phi}, which has one element, namely 0; 2 could be defined to be {phi, {phi}}, which has two elements, namely 0 and 1; 3 could be defined to be {phi, {phi}, {phi, {phi}}}, which has three elements, namely 0, 1, and 2; etc.

What is the simplest way to find a set intersection?

The difficulty of finding the intersection of two sets depends on the definitions of the sets, so there is no "simplest way" that always works.

If at least one of the sets is finite and is defined by a list of its elements, like A = {13, 84, 103, 194, 264} then you can find the intersection of A with B by checking to see if each element of A is in B.

But if A is infinite that method doesn't work well in general because it may take too long.

Sometimes it is easy to find the intersection of A and B. For example, if A consists of all multiples of 2 and B consists of all multiples of 3, then the intersection of A and B consists of all multiples of 6, and more generally, if A consists of all multiples of m and B consists of all multiples of n, then the intersection of A and B consists of all multiples of mn, if m and n are relatively prime — that is, have no divisors in common. If they do have common divisors, then the intersection of A and B consists of all multiples of mn/d, where d is the greatest common divisor of A and B.

Finding the intersection of two sets A and B, as in the three statements above, involves two separate actions — making an education guess about the intersection, and proving that the guess is correct; I have not done the latter in the paragraph above. In such an example, you need to have an awareness of the context within which the two sets A and B are defined, in this case, number theory.

Thus, finding the intersection of two sets is context-dependent, so there cannot be a single "simplest method" of accomplishing this task in all cases.

Sometimes we can't even tell whether a particular item is in the intersection of two sets. Thus, for example, it is not known whether e+pi is a rational number. That can be restated (perhaps not fruitfully) as meaning that we can't tell whether e+pi is in the intersection of the real numbers and the rational numbers.

What are the total number of sets in the world?

The world is a big place, but it is after all finite. So the number of sets in the world is finite as well, although that number must be very large.

On the other hand, although mathematicians are concerned with the world in which we live, they operate in an imaginary world in which there are infinite sets of infinitely many sizes. In that world, there cannot be a set of all sets that one can measure and, in that world, one cannot answer the question of how many sets there are altogether.

If a set is an unordered collection, then why do we talk about permutations of sets in group theory?

Yes, a set is an unordered collection of objects. Mathematicians, however, are not particularly interested in a set with no structure.

Thus mathematicians discuss various algebraic structures that can be imposed on a set — for example, the group of permutations of {1,2,3,4}, the ring of integers, or the field of rational numbers.

Another kind of structure that can be imposed on a set is that of an ordering — for example, the set of integers can be ordered by the relation "less than or equal to" and the set of subsets of {1,2,3,4} can be ordered by the relation "is a subset of".

The first example is a total ordering, that is, for every two integers M and N, either M is less than or equal to N or N is less than or equal to M — and the only way both happen is if M = N. The second example is a partial ordering that is not a total ordering, since neither {1,2} nor {3,4} is a subset of the other.

Why does a set M contain the empty set? I'm thinking specifically of the fact that $M\setminus\emptyset=M$, which would imply $\emptyset\notin M$.

The problem arises because the English word "contains" has two meanings in mathematics, and you have to be careful to understand from the context which of those two meanings is intended.

We sometimes use "M contains A" to mean "A is an element of M," and we sometimes use "M contains A" to mean "A is a subset of M." Thus, when we say that every set contains the empty set, we mean that the empty set is a subset of every set, and when we say that the set of prime numbers contains 17, we mean that 17 is an element of the set of prime numbers P.

Now, of course, we should be more careful and use "is an element of" and "is a subset of" to avoid any misunderstandings … we should say, for example, that the empty set is a subset of every set M and that 17 is an element of the set of prime numbers.

However, we clearly don't do that consistently … perhaps because its easier to say "M contains the empty set" and "P contains 17" even while we are writing the inclusion symbol for the first statement and the element symbol for the second.

This confusion of meanings of "contains" is, not surprisingly, the source of many peoples' confusion about these concepts.

What is the cardinality of all functions from a set to another set?

If A and B are both finite, |A| = a and |B| = b, then if f is a function from A to B, there are b possible images under f for each element of A. By the Multiplication Principle of Counting, the total number of functions from A to B is

b x b x b x b x … x b

where b is multiplied by itself a times. Thus the total number of functions from A to B is b^a, that is |B|^|A|.

So it makes sense to use the same notation |B|^|A| to represent the cardinality of the set of all functions from one set A to another set B.

But what is the value of |B|^|A| if A and/or B is infinite?

Let's take a look at the simplest examples — where one of A and B consists of 2 elements, and the other consists of a countably infinite set, like the set N of natural numbers.

One of those is easy: If A is a 2-element set and B = N, then |N|^|2|, the number of functions from a 2-element set to N, is just countable, since it really the same as the set of all ordered pairs of natural numbers … if we use aleph-sub-zero to refer to the cardinality of the natural numbers, then (aleph-sub-zero)^2 = aleph-sub-zero.

On the other hand: if we think of the two element set B as {0,1}, then |2|^|N| is the set of all infinite sequences of 0's and 1's, which is not really different from the set of all infinite sequences of numbers between 0 and 9, which is the set of all infinite decimals between 0 and 1, and that has the same cardinality as the set of real numbers R.

So |2|^|N| = |R|. You might think that the simplest infinite exponent would be aleph-sub-one, and many mathematicians tried to prove that this was so, but they failed. The statement that the first uncountable infinite cardinal is the cardinality of the real numbers is called the continuum hypothesis, where the real number line is called the continuum.

The continuum hypothesis was never proved, but Kurt Godel proved in 1940 that it was consistent with the axioms of set theory. However, in 1963 Paul Cohen proved that the continuum hypothesis was independent of the axioms of set theory. His proof of independence was first publicly announced at Cohen's presentation on Independence Day, July 4, 1963 at Berkeley. I was there!

So we can provide notation that represents the cardinality of the set of all functions from A to B, but if either A or B is infinite, we really can't definitely say which cardinal number |B|^|A| actually is. In some models of set theory, it may have a certain value and in other models of set theory, it may have a different value.

Does there exist a circle not intersecting QxQ?

The question asks whether there is a circle in the coordinate plane that has no points both of whose coordinates are rational numbers, that is, ones that deserve to be called a rational points.

Here is a different proof that there are such circles, and that in fact most circles have this property.

Since the rational numbers are countable, the same is true of the points with both coordinates rational, that is, the rational points. On the other hand, the number of circles centered at any point, like the origin, is uncountable since any real number can be the length of the radius. However, since no two of them have any points in common, only a countable number of them can have rational points. So that there are relatively few circles centered at any point that contain rational points.

If it's any consolation, every rational point is on an uncountable number of circles.

Is the empty set an element of the natural numbers ℕ?

The simple answer is that the empty set is not an element of the set of natural numbers — since none of the natural numbers is the empty set — although the empty set is a subset of the natural numbers. I didn't have any natural numbers for lunch today — although 3 tastes very good with hot sauce — so the set of natural numbers I had for lunch today is the empty set. You may have meant to ask the question "Is the empty set a subset of the natural numbers?" and the answer to that is "yes" because the empty set is a subset of every set.

On the other hand, many mathematicians and logicians in the 20th century wanted to base all of mathematics on set theory and were not satisfied with the lack of formal definitions of the natural numbers. What, for example, is 3? Is it a noun, in which case we should be able to see one in a zoo, or is it an adjective, like three-ness?

One solution was to identify 0 with the empty set E. Then 1 could be the set {E} with one element, the empty set. Then 2 could be the set {E, {E}} with two elements, 0 and 1. Then 3 could be the set {E, {E}, {E,{E}}} with three elements, 0, 1, and 2. In general, n could be the set with n elements, namely, 0, 1, 2, … , n-1 — all of which could be specified using E and brackets.

This approach allows you to specify exactly what each natural number is — 3, for example, is {E, {E}, {E,{E}}} — and 0 is specified to be E.

From this perspective, the empty set is 0 and 0 is a natural number. So, from this perspective, the answer to your question is yes, the empty set is a natural number.

However, this perspective is, by and large, ignored nowadays, when most mathematicians are not so interested in grounding mathematics in set theory.

So the safe answer is the simple answer with which I began, namely, the empty set is not a natural number.

In transfinite arithmetic, what exactly is the distinction between the Aleph numbers ($\aleph_0$, $\aleph_1$, etc.) and the Omegas (ω_0, ω_1, etc.)? Do they correspond?

In a nutshell, the Aleph numbers refer to the size (or cardinality) of a set whereas the Omega numbers refer to the ordering of a set.

For a finite set, say with 6 elements, there is essentially only one way of ordering the set — there is a first element, and a second element, … , and finally a sixth element. Of course, you can list the six elements in 6! different ways, but all of those orderings are essentially the same.

When you look at countable sets (those of size Aleph-zero), there are many different ways of ordering the set. For example, we can list its elements by counting them all out using the natural numbers. That seems like a natural way of counting out a countable set.

But there are other ways of listing a countable set. For example, we can list its elements by counting them out as we might the integers, so that the elements of the countable set are labeled using all the integers:

… -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, …

Or we might use a 1–1 correspondence between the elements of the countable set and the rational numbers to order the set like the rational numbers.

One property of the natural way of ordering a countable set is that every non-empty subset has a least element. This is called the well-ordering property, which for countable sets is equivalent to mathematical induction.

If we ordered the countable set like the integers, then the elements of the set that corresponded to the negative integers would be a non-empty subset that has no least element, violating the well-ordering property. Similarly, if we ordered the countable set like the rational numbers, then the elements of the set that corresponded to the positive rational numbers would be a non-empty subset that has no least element, again violating the well-ordering property.

You might think that the ordering of the natural numbers is the only ordering of the set that satisfies the well-ordering property. Not so. For example, here is an ordering of the natural numbers that is well-ordered but is not the same as the ordering of the natural numbers:

1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, … 5

Students frequently object to this ordering of the natural numbers because, they say, 5 belongs between 4 and 6, not out there at the end, beyond all the numbers. That is of course true for the natural ordering of the natural numbers, but mathematicians can order things however they want, and I have chosen to put 5 out at the end. (You should prove that this ordering is a well ordering.)

The natural ordering of the natural numbers is denoted Omega-zero, and this ordering would therefore be called Omega-zero + 1. Here is another ordering of the natural numbers:

1, 3, 5, 7, 9, 11, 13, … 2, 4, 6, 8, 10, 12, …

where all the even numbers come after all the odd numbers, but the evens and odds are both in their natural order. (You should prove that this ordering is also a well ordering.) This is different from Omega-zero + 1. Indeed, it would be denoted

Omega-zero + Omega-zero.

Exercise: Construct well-orderings of the natural numbers whose orderings would be

Omega-zero + Omega-zero + Omega-zero

Omega-zero + Omega-zero + Omega-zero + Omega-zero

Omega-zero + Omega-zero + Omega-zero + Omega-zero + Omega-zero

Etc., and even one whose ordering would be

Omega-zero + Omega-zero + Omega-zero + Omega-zero +Omega-zero + …

What distinguishes Omega-zero from all the other well-orderings of a countable set?

It is the "smallest" of all the well-orderings, in that every well-ordering of a countable set starts with a sequence of elements that matches up with Omega-zero.

Now we are ready to move on to Omega-one!

Omega-one is the smallest of all of the well-orderings of a set of cardinality Aleph-one.

How do we know that a set of cardinality Aleph-one can we well-ordered?

Well, we really don't.

However, if we assume the Axiom of Choice, then every set can be well-ordered so, in that system of set theory, there really is an Aleph-one.

If you want to find out more about all of this, find my book "Linear Orderings," which is available free online because it was written so long ago that the copyright has expired.

Why is the cardinality of a set's power set 2 raised to the cardinality of that set?

The power set P(X) of a set X is the set of all subsets of X.

As others have explained, when X is a finite set with n elements, the power set P(X) of X has 2^n elements — because to determine any particular subset of X, you have to make a yes-no choice for each of the elements of X. Since there are two possibilities for any element of X ("yes" or "no"), by the Multiplication Principle of Counting there are 2*2*2*…*2 (with n 2s) different ways of making these n choices. That means that if X has n elements, then P(X) has 2^n elements. That is why P(X) is called the "power set" of X.

If X is an infinite set, then again to determine any particular subset of X, you have to make a yes-no choice for each of the elements of X. How many elements are there in P(X)? By analogy, the number of elements of the power set P(X) is denoted 2^|X|, where we are using |X| to stand for the cardinality of X.

There is a difference between the use of this exponentiation notation for finite sets and the use of the same notation for infinite sets. In the case of finite sets, this exponentiation is part of ordinary arithmetic and is useful in a variety of contexts. In the case of infinite sets, this exponentiation is no more than a mathematical convention — we agree to refer to the cardinality of power sets using exponential notation, but this is only by analogy; we don't really work with this exponentiation as we do with exponentiation of whole numbers or real numbers.

For example, what is $3^{|X|}$ and how does it relate to $2^{|X|}$?

We can think of P(X) as the set of all functions from X to {0,1} — or {yes, no} — so we can imagine a set Q(X) as the set of all functions from X to {0,1,2}, and think of its cardinality as $3^{|X|}$.

If X is finite and has n elements, then P(X) actually has 2^n elements and Q(X) actually has 3^n elements — many more elements that P(X).

If X is infinite, however, then $2^{|X|} = 3^{|X|}$, that is to say, there is a 1–1 correspondence between the elements of P(X) and the elements of Q(X). That is, the infinite version of exponentiation doesn't obey the rules that we expect exponentiation to obey.

How do you demonstrate $|A|<|P(A)|$?

To clarify the question, if A is any set, then P(A) denotes the set of all subsets of A, and if B is any set then |B| denotes the cardinality of B, that is the number of elements in B.

If A is a finite set with n elements and you want to construct a subset of A, then for each element of A, there are two possibilities — either it will be in the subset of A that you are constructing or it it won't be in that subset. Since there are 2 choices for each element of A, and A has n elements, there are 2^n possible ways of constructing a subset of A by the Multiplication Principle of Counting. That is $|P(A)| = 2^n$. That is why P(A) is called the "power set of A."

If A and B are infinite sets, then we say that $|A| = |B|$ if there is a one-to-one correspondence between the elements of A and the elements of B. Thus, for example, the function f defined by f(a) =2a establishes a 1–1 correspondence between the integers and the even integers. This is paradoxical because there are lots of integers that are not even, so it would seem that there are more integers than there are even integers. According to this definition, however, they have the same size, the same cardinality. Though paradoxical, this definition of when $|A| = |B|$ is a very useful one, and is accepted in the mathematical world.

If A and B are infinite sets, then we say that $|A| < |B|$ is there is a one-to-one correspondence between the elements of A and a subset of B, but there is no one-to-one correspondence between A and B.

There certainly is a one-to-one correspondence between A and a subset of P(A), since we can associate with each element a of A the element {a} of P(A), the subset of A that has only one element, namely, a. In other words A is in one-to-one correspondence with the 1-element subsets of A.

So to answer the question we need to show that there is no one-to-one correspondence between A and P(A).

Suppose that there is a one-to-one correspondence h between A and P(A). What we will do is show that there is a subset D of A which is not the image h(a) for any a in A — in other words, D has been omitted from the 1–1 correspondence, contrary to hypothesis.

This is easy to do. Let D consist of those elements a of A that are not elements of h(a). That is, if a is an element of h(a), then a is not in D, whereas if a is not an element of h(a), then a is in D.

Because of this definition, D is different from h(a) for every a in A, since exactly one of them contains a. Thus D has been omitted from the 1–1 correspondence h. That means that h is not a 1–1 correspondence, contrary to assumption.

That is, there is no 1–1 correspondence between A and P(A), so that $|A| < |P(A)|$.

This argument, referred to as a diagonal argument (which is why I chose the name D for the set above), is due to Georg Cantor, the 19th century mathematician who essentially invented set theory.

Why is the empty set considered a set?

If you think of a set as a well-defined collection of objects, then the collection of human beings with five legs is certainly well- defined and should be considered a set, even though that collection has no elements.

If I have a bowl of apples on my table, the collection of apples in that bowl is a set. Even though, one by one, each of the apples is eaten, at each moment of time, the collection of apples in the bowl forms a well-defined set, which may at a certain time have no elements.

Just as we use 0 to represent the number of apples remaining in the bowl after all of the apples have been removed, so too we use the empty set to represent the set of apples remaining in the bowl after all the apples have been removed.

if you have difficulty grasping the existence of the empty set, you will have even more difficulty grasping the uniqueness of the empty set. That is, the set of apples in the bowl after all the apples have been removed is identical to the set of five-legged human beings!

That is the case because two sets are mathematically defined to be equal if every element of one is also an element of the the other, and the reverse. In this situation, every five-legged human being is in my apple-less bowl, and every apple in my apple-less bowl is a five-legged human being.

You may find this rather odd, and argue that a set of apples can’t be the same as a set of oranges. In that case you might adopt a philosophical perspective of **intentionality**, that is, two sets are equal if their descriptions are equivalent.

Mathematicians, on the other hand, adopt an **extensional** perspective and say that two sets are equal if they have equal extent, that is, have the same elements, even if the descriptions of the sets are quite different. That is why mathematicians refer to **the** empty set, because from an extensional perspective, there can be only one empty set.

Is it possible to build an uncountable set of sentences in Danish?

No. The set of sentences in Danish, or in any language, is countable. This answer is based on two assumptions about the language. First that the of letters in its alphabet is finite or countable and, second, that any sentence in the language consists of a finite sequence of letters in the alphabet.

For example, if the alphabet consists of the integers from 0 to 9, then the sentences would correspond to the finite decimals. One sentence in this language would be .5386724369801, a 13-letter sentence ending in 1.

The number of such finite sequences, that is, the number of sentences in this language, is countable. You can, in fact, enumerate all one letter decimals (there are ten of them), then all two letter decimals (there are 100 of them), etc.

If, however, sentences are allowed to run on forever, that is, can be infinite, then there would an uncountable set of sentences in that language. I don't think that's allowed in Danish, although we sometimes encounter people in any language who seem to go on forever.

Returning to our previous example, if you allowed infinite sequences of the integers from 0 to 9, then you would get all decimal numbers between 0 and 1, which Cantor showed was uncountable.

What is the difference between a countable and an uncountable set?

The simple but profound definition is that a set is countable if its elements can be matched up with the natural numbers, also referred to as the counting numbers, that is, the infinite set 1, 2, 3, 4, 5, … A set is uncountable if it is not countable.

A more formal definition is that a set is countable if there is a 1–1 correspondence between its elements and the natural numbers. For example, the set of positive even numbers is countable, since the function f defined by $f(x) = 2x$ is a 1–1 correspondence between the natural numbers and the positive even numbers.

Similarly, the set of powers of 2 is countable, since the function $g(x) = 2^x$ is a 1–1 correspondence between the natural numbers and the powers of 2. (That's not quite true, but I'll leave it to you to fix the error.)

Similarly, the set of all even numbers is countable , since the function h defined by cases, $h(2x) = 2x$ and $h(2x-1) = -2(x -1)$, is a 1–1 correspondence between the natural numbers and the set of all even numbers.

Perhaps surprisingly, the set of all rational numbers is also countable. For the positive rational numbers, make an infinite table with the natural numbers at both the top and at the left side, and put in each box the number above it divided by the number to its left. Every positive rational number appears in this table … indeed, it appears many times. Now count the entries (omitting duplicates) in a zig-zag fashion, starting with 1/1, 2/1, 1/2, 1/3, 2/2 (skip), 3/1, 4/1, 3/2, 2/3, 1/4, 1/5, … This gives a 1–1 correspondence between the natural numbers and the positive rational numbers.

The real numbers are uncountable, as are the real numbers between 0 and 1, demonstrated by Georg Cantor in this famous "diagonal argument."

For suppose that the real numbers between 0 and 1 were countable, we could line up all the decimals in a list, R-1, R-2, R-3, R-4, … .

Then we could define a real number R between 0 and 1 as follows:

The first decimal place of R is the the first decimal place of R-1 plus 1;

The second decimal place of R is the second decimal place of R-2 plus 1;

The third decimal place of R is the third decimal place of R-3 plus 1;

etc.

In general, the n'th decimal place of R is the n'th decimal place of R-n plus 1.

Note that if the n'th decimal place of R-n is 9, then the n'th decimal place of R is 0.

Now R is not equal to any R-n, since it differs from R-n in the n'th place.

Therefore R is not on the list. But the list contains all real numbers between 0 and 1, so R must be on the list. That's a contradiction, which proves that the real numbers between 0 and 1 is uncountable.