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A NOTE ON A THEOREM OF VAUGHT

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In [2] Vaught showed that if T is a complete theory formalized in the first-order predicate calculus, then it is not possible for T to have exactly (up to isomorphism) two countable models. In this note we extend his methods to obtain a theorem which implies the above.

First some definitions. We define $F_n(T)$ to be the set of well-formed formulas (wffs) in the language of T whose free variables are among $x_1, x_2, \cdots, x_n$. An n-type of T is a maximal consistent set of wffs of $F_n(T)$; equivalently, a subset P of $F_n(T)$ is an n-type of T if there is a model M of T and elements $a_1, a_2, \cdots, a_n$ of M such that $M \models \phi(a_1, a_2, \cdots, a_n)$ for every $\phi \in P$. In the latter case we say that $\langle a_1, a_2, \cdots, a_n\rangle$ realizes P in M. Every set of wffs of $F_n(T)$ which is consistent with T can be extended to an n-type of T.

By the completeness theorem every n-type of T is realized in some countable model M of T. If every n-type of T is realized in M, for every natural number n, we say that M is weakly-saturated. The model M of T is said to be saturated if, in addition, whenever P_n is an n-type of T, P_{n+1} is an $n+1$-type of T which contains P_n, and $\langle a_1, a_2, \cdots, a_n\rangle$ realizes P_n in M, then there is an a_{n+1} in M such that $\langle a_1, a_2, \cdots, a_n, a_{n+1}\rangle$ realizes P_{n+1} in M. A saturated model is also homogeneous—that is, if $\langle a_1, a_2, \cdots, a_n\rangle$ and $\langle b_1, b_2, \cdots, b_n\rangle$ are two n-tuples of elements of M which realize the same n-type of T, then there is an automorphism of M which sends each a_i to the corresponding b_i. In the paper mentioned above, Vaught also showed that if for each n there are at most a countable number of n-types of T, then T has a unique (up to isomorphism) countable saturated model. Note also that in this paper reference is made to the following theorem of Engeler, Ryll-Nardzewski [1] and Svenonius: Call two wffs of $F_n(T)$ equivalent with respect to T if, in every model of T, any n-tuple which satisfies one wff also satisfies the other; then a complete theory T has exactly one (up to isomorphism) countable model if and only if each $F_n(T)$ contains but a finite number of wffs which are inequivalent with respect to T.

Theorem. *Let T be a complete theory which has a finite number k (up to isomorphism) of countable models. If $k > 1$ then T has a weakly saturated model which is not saturated.*

Proof. Since T has only a finite number of countable models, it can have only a countable number of n-types for each n. Hence T has a unique saturated model M. Let $M_1, \cdots, M_{k-1}$ be the nonsaturated countable models of T. If none is weakly saturated then for each i there is an n_i-type P_i which is not realized in M_i. We introduce the following notation: If $\phi \in F_n(T)$ then ϕ^m is the wff

$$\phi(x_{m+1}, x_{m+2}, \cdots, x_{m+n}) \in F_{m+n}(T).$$

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Let $P_1^* = P_1$, let $P_2^* = \{\phi^{n_1} \mid \phi \in P_2\}$, $P_3^* = \{\phi^{n_1+n_2} \mid \phi \in P_3\}, \ldots$ and let $P_{k-1}^* = \{\phi^{n_1+n_2+\cdots+n_{k-2}} \mid \phi \in P_{k-1}\}$.

Then $P^* = \bigcup_{i=1}^{k-1} P_i^*$ is a consistent set of wffs of $F_n(T)$ where $n = \sum_{i=1}^{k-1} n_i$, hence can be extended to an n-type P of T. It is clear that P cannot be realized in any M_i since otherwise P_i would be realized in that M_i. On the other hand M is saturated so P is realized in M.

We now show that this situation is impossible—i.e. that if in a complete theory there is an n-type which is realized only in the saturated model, then the theory can have only one countable model altogether.

Extend the language of T by adding new constant symbols $e_1, e_2, \cdots, e_n$ and let $T' = T \cup \{\phi(e_1, \cdots, e_n) \mid \phi(x_1, \cdots, x_n) \in P\}$. Then a countable model of T' is a model of T which has distinguished in it particular elements $a_1, \cdots, a_n$ which realize P. Since the only model of T in which P is realized is M and since M is homogeneous, we may conclude that any two countable models of T' are isomorphic. Hence, by the theorem in [1] quoted above, for each n there are but a finite number of inequivalent elements of $F_n(T')$. In particular, with respect to T', the wffs of $F_n(T)$ fall into a finite number of equivalence classes. But if two wffs of $F_n(T)$ are equivalent with respect to T' then they are already equivalent with respect to T. Hence, again using the theorem in [1], any two countable models of T are isomorphic, contrary to the hypothesis of our theorem. This completes the proof.

Since a weakly-saturated model which is not saturated cannot be a prime model, this result implies the theorem of Vaught quoted above.

REFERENCES

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