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On $\aleph_0$-Categorical Abelian by Finite Groups

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1. INTRODUCTION

A countable group G is $\aleph_0$-categorical if it can be characterized, up to isomorphism, within the class of countable groups, by its first-order properties. Among the classes of groups investigated for $\aleph_0$-categoricity by Rosenstein [9] was the class of groups G which have a normal Abelian subgroup A of finite exponent and finite index; we refer to such groups as Abelian by finite groups.

The nonlogician may wish to refer to the Introduction of [9] for a detailed description of what $\aleph_0$-categoricity means logically. Alternatively, he may prefer to use the following algebraic condition, which, by the Basic Theorem on $\aleph_0$-categoricity (shown independently by Engeler, Ryll–Nardzewski, and Svenonius), is equivalent to the logical condition of $\aleph_0$-categoricity:

G is $\aleph_0$-categorical if for each n the number of n-orbits of G is finite,

(where two n-tuples $\langle a_1, a_2, \ldots, a_n\rangle$ and $\langle b_1, b_2, \ldots, b_n\rangle$ of elements of G are in the same n-orbit if there is an automorphism f of G satisfying $f(a_i) = b_i$ for each i). See [9] also for a discussion of other articles dealing with $\aleph_0$-categorical structures.

If the exponent of A is square-free and G/A is cyclic then, as is shown in [9], G must be $\aleph_0$-categorical. In this paper we deal with the case where the exponent of A is still square-free but G/A is not necessarily cyclic. In particular we consider exhaustively the case where A has exponent 2 and $G/A \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, giving necessary and sufficient conditions for such a group to be $\aleph_0$-categorical.

The result in [9] quoted above can be extended so that if the exponent of A is square-free and all Sylow subgroups of A are cyclic, then G must be $\aleph_0$-categorical. Combining these two conclusions, we arrive at necessary and sufficient conditions for a group G to be $\aleph_0$-categorical if the exponent of A is square-free and all Sylow subgroups of G/A are either cyclic or $\mathbb{Z}_2 \times \mathbb{Z}_2$. (See Theorem 17.)

We turn now to the specific case mentioned above. Given that $G/A \simeq$

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$\mathbb{Z}_2 \times \mathbb{Z}_2$ and that A has exponent 2, we will construe A as an S-module, where S (the group ring) we now define explicitly. Given $G = \langle A, d, e \rangle$, we can consider A as a vector space over F_2 and we can consider d and e as linear transformations δ and ϵ of A defined by

$$\delta(a) = d^{-1}ad, \qquad \epsilon(a) = e^{-1}ae$$

for all $a \in A$. These two linear transformations generate a ring $S = F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$ of linear transformations of A, whose action on A determines an S-module structure on A. (More generally, if A has exponent p^k, then A can be considered an R-module, where $R = Z_{p^k}$. Moreover, if $H = G/A$, then A can be considered an $R[H]$-module, where $R[H]$ is the group ring of H over R. This group ring consists of all formal finite sums $\sum r_i \gamma_i$ where each $r_i \in R$ and each $\gamma_i \in H$. For further information, see, for example, Curtis and Reiner [5].)

The question of which Abelian by finite groups G, where A has exponent 2 and $G/A \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, are $\aleph_0$-categorical can be transformed into the question of which $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$-modules A are $\aleph_0$-categorical. (When we speak of $\aleph_0$-categoricity for R-modules, we always mean that R is a fixed finite ring and that constant symbols for the elements of R are introduced into the language of groups to convert it into the language particular to R-modules.) Indeed, we shall show (Theorem 17) that in general, with G, A, R, and H as above, G is $\aleph_0$-categorical as a group if and only if A is $\aleph_0$-categorical as a $R[H]$-module. Since $\aleph_0$-categoricity is preserved, switching categories is sensible.

In the case of $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$-modules, we are then able to determine explicitly which ones are $\aleph_0$-categorical. This is made possible because of a further reduction. Let $x = 1 + \delta$ and $y = 1 + \epsilon$ denote two specific elements of $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$ and define a relation R on A by specifying that

$$\langle v, w \rangle \in R \quad \text{iff for some} \quad a \in A,$$
$$x(a) = v \quad \text{and} \quad y(a) = w.$$

Informally, $\langle v, w \rangle \in R$ iff $w = yx^{-1}(v)$.

We will first analyze structures of the form $\langle A; R \rangle$ where R is a linear relation on the vector space A and obtain (Theorem 1) an explicit criterion for $\aleph_0$-categoricity of these linear relations (Sections 3–8). We will then show that given an $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$-module A, it is $\aleph_0$-categorical if and only if the associated linear relation $\langle A; R \rangle$ is $\aleph_0$-categorical (Section 9). This will give us an *explicit algebraic* criterion for $\aleph_0$-categoricity of $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$-modules (Theorem 13). We will then prove the general result connecting the $\aleph_0$-categoricity of the group G with the $\aleph_0$-categoricity of its associated $R[H]$-module and thereby obtain an explicit algebraic criterion for $\aleph_0$-categoricity of the corresponding class of Abelian by finite groups.

Baur [2] has given a general classification of all $\aleph_0$-categorical modules. His analysis thus gives another criterion for $\aleph_0$-categoricity of $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$-modules; although necessarily our explicit criterion must be equivalent to Baur's criterion, there does not seem to be a direct route from one to the other. (See also [1].)

In the concluding sections of this paper, we will present a number of general results concerning $\aleph_0$-categoricity of modules. In particular, we will show that, in the classification of Abelian by finite groups, there is no serious loss of generality in considering only the case where A has exponent a power p^k of a prime p and $H = G/A$ is a p-group.

The notation and terminology described in this introduction will be maintained throughout the paper. Note in particular that S always denotes the group ring $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$.

2. Examples of Non-$\aleph_0$-categoricity

In this section, we present the basic types of Abelian by finite groups G for which A has exponent 2 and $G/A \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, but which are not $\aleph_0$-categorical.

It will be useful to switch categories and think of A as an S-module, as discussed in the Introduction. When we think of A as a vector space, we will also refer to it as V. Decompose V into two infinite-dimensional F_2-spaces, $V = U \oplus W$, and let θ be a fixed isomorphism $\theta\colon U \simeq W$. Given an endomorphism T of W, we turn V into an S-module by defining the action of $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$ on V as follows:

$$\begin{aligned} xw &= yw = 0 && \text{for } w \in W, \\ xu &= \theta(u) && \text{for } u \in U, \\ yu &= T\theta(u) && \text{for } u \in U. \end{aligned}$$

In the corresponding group, A is decomposed into $B \oplus C$. Since $x = 1 + \delta$, where $\delta(c) = d^{-1}cd$, the equation $xw = 0$ becomes $d^{-1}cd = c$; and similarly $yw = 0$ becomes $e^{-1}ce = c$. On the other hand $xu = \theta(u)$ becomes $b(d^{-1}bd) = \theta(b)$; finally $yu = T\theta(u)$ becomes $b(e^{-1}be) = T\theta(b)$. Thus the corresponding group can be described as $G = \langle B, C, d, e\rangle$ with relations that guarantee that $B + C$ is Abelian of exponent 2, that $d^2 = 1$, $e^2 = 1$, and that

$$\begin{aligned} d^{-1}cd &= c, & e^{-1}ce &= c; \\ d^{-1}bd &= b\theta(b), & e^{-1}be &= bT\theta(b). \end{aligned}$$

Now it is easily seen that in V we can define W as the range of x and we can define T on W as yx^{-1} (more precisely, given $w \in W$, choose u so that $xu = w$

and define $Tw = yu$, the value being independent of the u chosen). Hence $\langle W; T\rangle$ is definable in V. Thus V is $\aleph_0$-categorical as an S-module only if $\langle W; T\rangle$ is $\aleph_0$-categorical as a vector space equipped with an endomorphism.

As we will see in Section 9, the archetypical non-$\aleph_0$-categorical structures $\langle W; T\rangle$ have the following forms:

1. For each $w \in W$ there is an n such that $T^n w = 0$, but there is no uniform bound on n. We call such a map *nil* but not *nilpotent*.

2. T is a surjective shift operator; i.e., for a basis $\{w_i \mid i \in \mathbb{Z}\}$ of W we have $Tw_i = w_{i+1}$.

3. T is a nonsurjective shift operator; i.e., for a basis $\{w_i \mid i \in \mathbb{N}\}$ of W we have $Tw_i = w_{i+1}$.

The non-$\aleph_0$-categorical groups G_2 and G_3 corresponding to Examples 2 and 3 can be described as follows:

$$G_2\colon \langle b_i, c_i \mid i \in \mathbb{Z}\rangle \cup \langle d, e\rangle,$$
$$G_3\colon \langle b_i, c_i \mid i \in \mathbb{N}\rangle \cup \langle d, e\rangle,$$

where in each case $\theta(b_i) = c_i$ and the relations are

$$\begin{aligned} b_ib_j &= b_jb_i, & d^{-1}c_id &= e^{-1}c_ie = c_i,\\ c_ic_j &= c_jc_i, & d^{-1}b_id &= b_ic_i,\\ b_ic_j &= c_jb_i, & e^{-1}b_ie &= b_ic_{i+1}, \end{aligned}$$
$$b_i^2 = c_i^2 = d^2 = e^2 = 1.$$

There is a real difference between Examples 2 and 3. The failure of $\aleph_0$-categoricity in Example 3 arises already from the definability of a great number of subspaces T^nW (or subgroups $\langle c_i \mid i \geqslant n\rangle$); model-theoretically, there are many 1-types. In Example 2, there is no such class of subspaces; the failure of $\aleph_0$-categoricity depends on the existence of many 2-types. On the level of algebra, this difference manifests itself in the following way. Assume for simplicity that T is a monomorphism, as in Examples 2 and 3 (although we will also deal with the more general case). Define

$$I = \{w \in W \mid T^n w \text{ and } T^{-n}w \text{ are defined for all } n\}.$$

Then T acts as an automorphism of I. In Example 2 it is the structure $\langle I; T\rangle$ which fails to be $\aleph_0$-categorical. In Example 3 it is the gradual vanishing of $I = \bigcap_n (T^nW \cap T^{-n}W) = (0)$ that ruins $\aleph_0$-categoricity.

We will see in Sections 3 and 9 that these three examples do in fact exhaust all possible sources of non-$\aleph_0$-categoricity in extensions of Abelian groups of exponent 2 by $\mathbb{Z}_2 \times \mathbb{Z}_2$.

3. Linear Relations and $\aleph_0$-Categoricity

In Sections 3–8, we will completely classify $\aleph_0$-categorical structures of the form $\langle V; R\rangle$, where V is a vector space (over a finite field) and R is a linear relation on V, i.e., a linear subspace $R \subseteq V \times V$. When R is simply an endomorphism of V, the analysis becomes significantly simpler. However, we do need to analyze a *general* linear relation, as is evident from the discussion in Sections 1 and 2. Recall that we intend to pass from a group $G = \langle A, d, e\rangle$ to a linear relation $\langle A; R\rangle$ via the S-module structure on A. Thus

$$\langle v, w\rangle \in R \quad \text{iff for some} \quad a \in A,$$
$$xa = v \quad \text{and} \quad ya = w,$$

where $x = 1 + \delta$ and $y = 1 + \epsilon$ so that $xa = ad^{-1}ad$ and $ya = ae^{-1}ae$. (In the examples considered in Section 2, the relation R, defined in exactly the same way, was an endomorphism T of the subspace W.) In Section 9 we will see how to reconstruct the S-module A from the linear relation $\langle A; R\rangle$ and in Section 10 we will see how to reconstruct the group G from the S-module A.

The analysis of an $\aleph_0$-categorical endomorphism reduces essentially to the analysis of a nilpotent map and an automorphism. More generally, we will introduce a class of "nilpotent" linear relations, and divide the analysis of $\aleph_0$-categorical linear relations into the analysis of nilpotent linear relations and automorphisms.

The statement of our main theorem depends on a preliminary analysis of general linear relations. Let $\langle V; R\rangle$ be a given linear relation. Define

$$R^n 0 = \{v \in V \mid \text{There are } x_1, \ldots, x_n \text{ such that } vRx_1, x_1Rx_2, \ldots, x_{n-1}Rx_n \text{ and } x_n = 0\}.$$

$$0R^n = \{v \in V \mid \text{There are } x_1, \ldots, x_n \text{ such that } 0 = x_1 \text{ and } x_1Rx_2, \ldots, x_{n-1}Rx_n, x_nRv\}.$$

For $m < n$ note that $R^m 0 \subseteq R^n 0$ and $0R^m \subseteq 0R^n$. Set $Z = \bigcup_n R^n 0$, $Z' = \bigcup_n 0R^n$. Next let

$$I = \{v \in V\colon \text{there is some doubly infinite sequence } \{v_i\} \subseteq V \text{ with } v_0 = v \text{ and } v_iRv_{i+1} \text{ for all } i \in \mathbb{Z}\}.$$

Notice that $Z \cap Z' \subseteq I$ (the desired sequence $\{v_i\}$ may be taken to consist largely of 0). R induces a linear relation R' on the subspace I which in turn induces a linear relation $\bar{R}$ on $I/Z \cap Z'$. [Here $(a + (Z \cap Z'))\,\bar{R}(b + (Z \cap Z'))$ iff for some $u \in (a + (Z \cap Z'))$, $v \in (b + (Z \cap Z'))$ we have $uR'v$.]

Theorem 1. *With the above notation $\langle V; R\rangle$ is $\aleph_0$-categorical iff:*

1. *For some* $n \geqslant 0$, $Z = R^n 0$ *and* $Z' = 0R^n$.
2. $\bar{R}$ *is an automorphism of* $I/Z \cap Z'$ *and* $\langle I/Z \cap Z'; \bar{R}\rangle$ *is* $\aleph_0$*-categorical.*
3. *For some integer* $n \geqslant 0$

$$I = \{v \in V\text{: there is a sequence } v_{-n}, v_{-n+1}, \ldots, v_{n-1}, v_n \text{ with } v = v_0 \text{ and } v_i R v_{i+1} \text{ for } -n \leqslant i < n\}.$$

When $V = Z + Z'$ it develops that condition 1 is necessary and sufficient for the $\aleph_0$-categoricity of $\langle V; R\rangle$. (Condition 2 trivializes and condition 1 implies condition 3 in this case, as will become evident.) We will call R *nil* iff $V = Z + Z'$, and *nilpotent* iff R is nil and satisfies condition 1. In the next few sections we treat the following topics:

4. Necessity of conditions 1–3 for $\aleph_0$-categoricity of $\langle V; R\rangle$.
5. Finite lattices of vector spaces.
6. Nilpotent linear relations.
7. Monomorphisms.
8. Sufficiency of conditions 1–3 for $\aleph_0$-categoricity of $\langle V; R\rangle$.

4. Necessity of Conditions 1–3 for $\aleph_0$-Categoricity of $\langle V; R\rangle$

Using the Basic Theorem on $\aleph_0$-categoricity, if $\langle V; R\rangle$ is $\aleph_0$-categorical, then condition 1 evidently holds. Condition 3 is also a straightforward consequence of the $\aleph_0$-categoricity of $\langle V; R\rangle$. Now conditions 1 and 3 imply that $\langle I/Z \cap Z'; \bar{R}\rangle$ is definable over $\langle V; R\rangle$ and is consequently $\aleph_0$-categorical if $\langle V; R\rangle$ is. We need therefore only verify that $\bar{R}$ is an automorphism of $I/Z \cap Z'$.

It is our intention to show that $\bar{R}$ is single-valued and 1–1; it is clear from the definition of I that $\bar{R}$ is everywhere defined and onto.

For any subspace W of R define

$$RW = \{v \in V \mid \text{for some } w \in W, vRw\}$$

and

$$WR = \{v \in V \mid \text{for some } w \in W, wRv\}.$$

Then $\bar{R}$ is single-valued iff $I \cap (Z \cap Z')R = Z \cap Z'$ and $\bar{R}$ is 1–1 iff $I \cap R(Z \cap Z') = Z \cap Z'$. By virtue of the symmetry present, we may confine ourselves to a demonstration of the latter identity.

It is easily seen that $Z \cap Z' \subseteq I \cap R(Z \cap Z')$. The converse inclusion is a consequence of the following two useful relations:

(i) $R(Z+Z') \subseteq Z+Z'$,
(ii) $I \cap (Z+Z') \subseteq Z \cap Z'$;

for, using these relations, we get $I \cap R(Z \cap Z') \subseteq I \cap R(Z+Z') \subseteq I \cap (Z+Z') \subseteq Z \cap Z'$. We will see that (i) is essentially a consequence of the definitions, while (ii) follows from condition 1 above.

Verification of (i). Suppose $xR(y_1+y_2)$ with $y_1 \in Z$, $y_2 \in Z'$. Choose $x' \in Z'$ such that $x'Ry_2$. Let $x'' = x - x'$ so that $x = x' + x''$, $x' \in Z$, and $x''Ry_1$. Then $x'' \in Z'$, so $x \in Z + Z'$.

Verification of (ii). We will give the verification that $I \cap (Z+Z') \subseteq Z$. That it is also included in Z' is proved symmetrically. Let $Z'_k = 0R^k$, so that $Z' = \bigcup_k Z'_k$. We will prove by downward induction on k:

$$(*) \qquad I \cap (Z+Z') \subseteq Z + Z'_k.$$

This is true for $k = n$ (since, by condition 1, $Z'_n = Z'$) and for $k = 1$ its truth will give the desired result by the following little argument: if $x \in I \cap (Z+Z')$ choose $y \in I$ so that xRy, apply the symmetric version $(Z+Z')R \subseteq Z+Z'$ of (i) to conclude that $y \in I \cap (Z+Z')$, and apply $(*)$ for $k=1$ to get $y \in Z + Z'_1$, so that we may write

$$y = z + z'_1 \qquad \text{with} \quad z \in Z, \quad 0Rz'_1.$$

Thus $x = (x-0)\, R(y - z'_1) = z$, proving $x \in Z$, as desired.

To proceed then with the proof of $(*)$, let us fix k ($1 \leqslant k < n$) and assume

$$(k+1) \qquad I \cap (Z+Z') \subseteq Z + Z'_{k+1}.$$

Fix $x \in I \cap (Z+Z')$, with a view toward proving that $x \in Z + Z'_k$.

First choose $y \in I$ so that xRy. Apply the symmetric version of (i) and $(k+1)$ to write $y = y_1 + y_2$ with $y_1 \in Z$, $y_2 \in Z'_{k+1}$. Fix $x' \in Z_k'$ such that $x'Ry_2$ and let $x'' = x - x'$. Then $x''Ry_1$ and $y_1 \in Z$, so $x'' \in Z$. We now have $x = x' + x''$, $x' \in Z'_k$, $x'' \in Z$, proving $x \in Z + Z'_k$.

Thus (ii) may be considered established. We have therefore completed the proof of the necessity of condition 2 above, and therewith the proof of the necessity of conditions 1–3. ▮

5. Finite Lattices of Vector Spaces

In this section we will discuss finite lattices $\mathscr{L}$ of subspaces of a vector space W. The method used here, that of embedding a finite distributive lattice in a Boolean algebra—referred to as "Booleanization"—is fundamental to our study of Abelian by finite groups.

Consider now the structure $\langle W; \mathscr{L} \rangle$ consisting of the space W with the elements of $\mathscr{L}$ as distinguished subspaces; assume that $\mathscr{L}$ is closed under the operations $X \cap Y$ and $X + Y$. We will refer to $\langle W; \mathscr{L} \rangle$ as a "finite lattice of vector spaces," despite the possibility that infinitely many subspaces of W may be definable in $\langle W; \mathscr{L} \rangle$; thus finite lattices of vector spaces, in this sense, need not be $\aleph_0$-categorical, even if the base field is assumed to be finite.

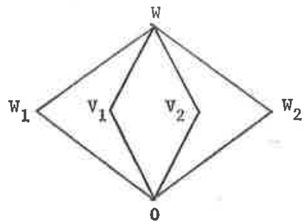


FIGURE 1

The basic example, which we now present, resembles those exploited by Baur [3]. Consider any finite lattice $\mathscr{L}$ of vector subspaces of W which contains four incomparable subspaces W_1, W_2, V_1, V_2 such that $W_i + V_j = W$ and $W_i \cap V_j = 0$ for all i and j. The simplest example of such a lattice is shown in Fig. 1. Now, for each j, V_j induces an isomorphism T_j from W_1 to W_2, defined by $T_j(w_1) = w_2$ iff $w_1 + w_2 \in V_j$. In particular $T = T_2^{-1}T_1$ is an automorphism of W_1. As is evident, the structure $\langle W_1 ; T \rangle$ is definable over $\langle W; \mathscr{L} \rangle$; thus if $\langle W_1 ; T \rangle$ is not $\aleph_0$-categorical, then neither is this "finite lattice" of vector spaces $\langle W; \mathscr{L} \rangle$. To make this last remark nonvacuous, we must show that it is possible for a finite lattice $\langle W; \mathscr{L} \rangle$ as described above to yield a non-$\aleph_0$-categorical $\langle W_1 ; T \rangle$. Of course much more is true: If T is any automorphism of a vector space W_1 which has no fixed points, then there is such a finite lattice $\langle W; \mathscr{L} \rangle$, described below, which induces $\langle W_1 ; T \rangle$; moreover, as will be clear later, non-$\aleph_0$-categorical $\langle W_1 ; T \rangle$ are far from rare. [Given $\langle W_1 ; T \rangle$, construct $\langle W; \mathscr{L} \rangle$ as follows: Take W_2 to be a vector space isomorphic to W_1 but disjoint from it. Let $W = W_1 \oplus W_2$ and let $\bar{w}$ denote

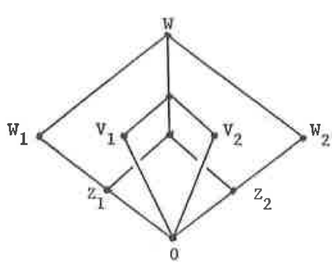


FIGURE 2

the element of W_2 corresponding to $w \in W_1$. Let $V_1 = \{w + \overline{Tw} \mid w \in W_1\}$ and let $V_2 = \{w + \bar{w} \mid w \in W_1\}$. Then, as is easily verified using the modular law (see below), the lattice $\mathscr{L}$ of subspaces of W generated by W_1, W_2, V_1, and V_2 is of the correct type (Fig. 2) and induces $\langle W_1; T\rangle$.] (The reader may wish to consider what happens if T is not assumed to have no fixed points.)

In this section, we will prove that a finite *distributive* lattice of vector spaces is $\aleph_0$-categorical. As the example above illustrates, not every lattice of vector spaces is distributive; however, every lattice of vector spaces does satisfy the modular law

$$a \cdot (b + c) = a \cdot b + c \qquad \text{if} \quad c \leqslant a.$$

The lattices of vector spaces which will arise in our analysis have the further property that each is a slight extension of a lattice which is generated by two chains. The fact that a modular lattice generated by two finite chains must be finite and distributive (see Birkhoff [4]) thus implies, in view of the above result, that each of our lattices of vector spaces will be $\aleph_0$-categorical.

In succeeding sections we intend to apply the proof, as well as the statement, of the following result, which is modeled on the proof of Grätzer [7] that a finite distributive lattice admits a canonical "Booleanization." (It will be evident that the proof applies to lattices of submodules of a semisimple module, as well as to lattices of vector spaces.)

THEOREM 2. *Let L be a finite distributive lattice of subspaces of the vector space W (over a finite field). Then $\langle W; L\rangle$ is $\aleph_0$-categorical.*

Proof. Given an element U of a finite lattice we let $\hat{U} = \bigcup\{V \mid V < U\}$. We say that U is join-irreducible if $\hat{U} < U$ or, equivalently, if U cannot be written as $U_1 \cup U_2$ where each $U_i < U$.

Let J be the set of join-irreducible elements U of L. For each $U \in J$ choose a space U^* so that $U = \hat{U} \oplus U^*$. Let $J^* = \{U^* \mid U \in J\}$ and let B be the lattice generated by J^*. We claim that B is a finite Boolean algebra containing L. Then the isomorphism type of $\langle W; B\rangle$ is completely determined by the dimensions of the atoms of B together with the dimension of $W - \bigcup\{U^* \mid U \in J\}$. Thus $\langle W; B\rangle$ is $\aleph_0$-categorical and so the same is true for $\langle W; L\rangle$.

To prove the claim, we argue first that $L \subseteq B$. Since L is finite and hence every element of L is a union of join-irreducibles, it suffices to show that $J \subseteq B$. We proceed by induction on the ordering of J. For atoms $U \in J$, $U = U^* \in B$. For nonatoms $U \in J$, $\hat{U}$ is a union of smaller join-irreducible elements, so that $\hat{U} \in B$ by induction hypothesis. Hence $U = \hat{U} \oplus U^*$ is in B. Hence $L \subseteq B$.

Notice next that the sum $\sum\{U^* \mid U \in J\}$ is direct, for if $\sum w_U = 0$ with each $w_U \in U^*$ and if we choose U to be a maximal element of J for which $w_U \neq 0$, then

$$w_U \in U \cap \sum\{U_1^* \mid U_1 \in J,\, U_1 \cap U \leqslant \hat{U}\} \leqslant \hat{U},$$

since L is distributive. But $w_U \in U^*$, so $w_U = 0$, a contradiction. It follows that B is a Boolean algebra with atoms $\{U^* \mid U \in J\}$. ▮

In the above argument the choice of the atoms U^* is completely arbitrary. The key point in the next four sections is this: If W carries additional structure, it may be possible to adapt the choice of the spaces U^* to reflect this structure.

6. Nilpotent Linear Relations

In this section we will prove that any nilpotent linear relation is $\aleph_0$-categorical, and, at the same time, fully elucidate the structure of such relations. To fix our notation, let $\langle V; R\rangle$ be a nilpotent linear relation. We let $Z_i = R^i 0$ and $Z'_j = 0R^j$ for each positive i and j. By the nilpotency assumption, there is an N and an N' such that $Z_n = Z_N$ for all $n \geqslant N$ and $Z'_n = Z'_{N'}$ for all $n \geqslant N'$. We may assume that $N \leqslant N'$, for otherwise we could replace R by R^{-1}.

Let L be the lattice generated by all of the subspaces $\{Z_i \mid i \leqslant N\} \cup \{Z'_j \mid j \leqslant N'\}$ of V. This lattice is necessarily a modular lattice, and, as noted earlier, since it is generated by two finite chains, is finite and distributive. Hence, by Theorem 2, the finite lattice $\langle V; L\rangle$ of vector spaces is $\aleph_0$-categorical. This conclusion is not sufficient for our purposes, however, since the action of R in the linear relation $\langle V; R\rangle$ is largely suppressed in the lattice $\langle V; L\rangle$ of vector spaces. Thus the result that $\langle V; L\rangle$ is $\aleph_0$-categorical does not yield the result that $\langle V; R\rangle$ is $\aleph_0$-categorical. To get the desired result, we will Booleanize the lattice $\langle V; L\rangle$ more carefully than was done in the proof of Theorem 2.

Before carrying out this Booleanization, we review the structure of the modular lattice $F(N, N')$ freely generated by two chains:

$$C: \quad c_1 < c_2 < \cdots < c_N ,$$
$$C': \quad c'_1 < c'_2 < \cdots < c'_{N'} .$$

Let $c_{ij} = c_i \cap c'_j$ for $i \leqslant N$, $j \leqslant N'$. As illustrations, we exhibit the lattices $F(2, 2)$ and $F(2, 3)$ in Figs. 3 and 4, respectively. As these diagrams illustrate, the join-irreducible elements are precisely the elements c_i, c'_j, and c_{ij}. In Birkhoff [4] the free lattices $F(N, N')$ are described in some detail, and, in particular, a "normal form" for elements of $F(N, N')$ is given: Every element of $F(N, N')$ is a union of the join-irreducible elements c_i, c'_j, and c_{ij}.

As we will see, the lattice L generated by the two chains $\{Z_i \mid 1 \leqslant i \leqslant N\} \cup \{Z'_j \mid 1 \leqslant j \leqslant N'\}$ cannot be freely generated by these chains. However, as L is a homomorphic image of $F(N, N')$, its join-irreducible elements lie among Z_i, Z'_j, and $Z_{ij} = Z_i \cap Z'_j$. (This is easily verified, remembering that any element of a finite lattice is a union of join-irreducible elements.)

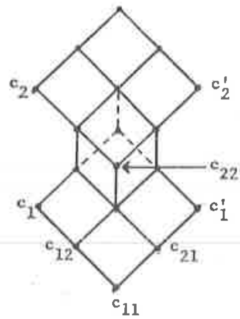


FIG. 3. $F(2, 2)$.

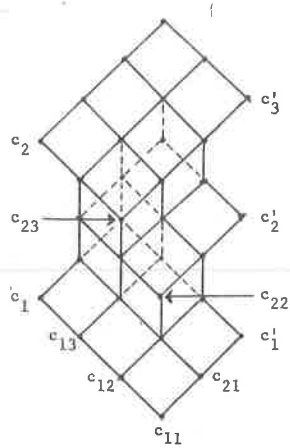


FIG. 4. $F(2, 3)$.

In the free lattice $F(N, N')$ on the chains C and C' we have the following identities:

(1) $\hat{c}_{i+1} = c_i \cup c_{i+1,N'}$;

(2) $\hat{c}'_{j+1} = c'_j \cup c_{N,j+1}$;

(3) $\hat{c}_{ij} = c_{i-1,j} \cup c_{i,j-1}$

(where, by convention, c_0 and c'_0, and hence also $c_{0,j}$ and $c_{i,0}$, are all 0).

Hence in the lattice L we have the following identities:

(1) $\hat{Z}_{i+1} = Z_i + Z_{i+1,N'}$;

(2) $\hat{Z}'_{j+1} = Z'_j + Z_{N,j+1}$;

(3) $\hat{Z}_{ij} = Z_{i-1,j} + Z_{i,j-1}$

(where, by convention, Z_0, Z'_0, $Z_{0,j}$, and $Z_{i,0}$ are all 0).

Note that in writing these identities we do not presuppose that Z_{ij} (or Z_i or Z'_j) is actually join-irreducible. In fact, we will now show that L is a proper homomorphic image of $F(N, N')$ by proving that, under certain circumstances, $Z_{ij} = \hat{Z}_{ij}$. Then Z_{ij} will not be join-irreducible and from (3) we conclude that $Z_{ij} = Z_{i-1,j} \cup Z_{i,j-1}$.

LEMMA 3. *If $i + j > N + 1$, then $Z_{ij} = \hat{Z}_{ij}$.*

Proof. Recall that we have assumed that $N \leqslant N'$. Let us first suppose that $i, j > 1$. Fix $x \in Z_{ij}$. Then, by definition of Z_{ij}, there are elements $y_1, y_2, \ldots, y_{i-1}$ and $z_1, z_2, \ldots, z_{j-1}$ such that

$$(*) \qquad 0Rz_{j-1}R \cdots Rz_1RxRy_1R \cdots Ry_{i-1}R0.$$

By assumption $(N - i + 1) < j$ so z_{N-i+1} exists, and, by $(*)$, $z_{N-i+1} \in R^{N+1}0 = R^N0$. Thus there are $N - 1$ elements

$$t_1, \ldots, t_{N-i}, x_1, s_1, \ldots, s_{i-2}$$

satisfying

$$(**) \qquad z_{N-i+1}Rt_{N-i}R \cdots Rt_1Rx_1Rs_1R \cdots Rs_{i-2}R0.$$

Subtracting $(**)$ from $(*)$ yields

$$0R(z_{N-i} - t_{N-i})R \cdots R(z_1 - t_1)\, R(x - x_1)\, R(y_1 - s_1)R \cdots Ry_{i-1}R0.$$

Thus $(x - x_1) \in Z_{i,N-i+1} \subseteq Z_{i,j-1}$ (since $N - i + 1 < j$). Since $x_1 \in Z_{i-1,j}$, we can conclude that $x = (x - x_1) + x_1 \in Z_{i,j-1} + Z_{i-1,j}$.

For $i = 1$ or $j = 1$ the proof is similar. We state these two cases more explicitly:

$$\text{for} \quad i > N, \qquad Z_{i,1} = Z_{i-1,1},$$
$$\text{for} \quad j > N, \qquad Z_{1,j} = Z_{1,j-1}. \quad ▮$$

This lemma merited our attention because it makes explicit all relations which fail in $F(N, N')$ but hold necessarily in L. In other words, if we let $L(N, N')$ be $F(N, N')$ modulo the relations of this lemma, then in general L is a homomorphic image of $L(N, N')$, but under certain circumstances it will be precisely $L(N, N')$. This may be seen by examples. The reader can verify that the lattices pictured in Figs. 5 and 6 are $L(2, 2)$ and $L(2, 3)$, and that, in the example $\langle V; R\rangle$ given below, the lattice L of subspaces is precisely $L(2, 3)$. As this analysis is not at all essential for the sequel, however, we will not dwell on it. [Example: Let V be a vector space over F_2 generated by $\{a_1, a_2, a_3, b_1, b_2, x, y, z\}$ and let R be generated by $b_1Ra_1R0RxRyRz$, $0Rb_2Ra_2R0$, and $0Ra_3R0$.]

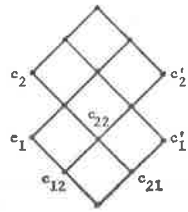


FIG. 5. $L(2, 2)$.

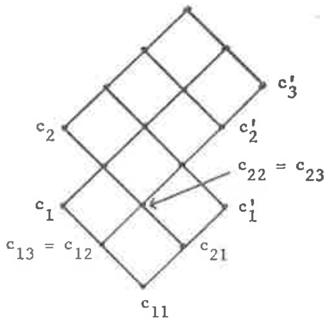


FIG. 6. $L(2, 3)$.

It is also useful to look at $L(3, 3)$, particularly with reference to Lemma 4 below. Fortunately, although $F(3, 3)$ is not nicely representable in three dimensions, its homomorphic image $L(3, 3)$ is, and has the representation shown in Fig. 7.

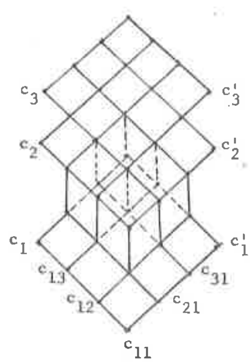


FIG 7. $L(3, 3)$.

In order to prove that all nilpotent linear relations are $\aleph_0$-categorical, we will Booleanize the lattice L; more precisely, we will embed L in a finite Boolean algebra B of subspaces of V in such a way that the action of R between atoms of B is easily described. We will be more precise momentarily, but it will be clear that the structure of $\langle V; R\rangle$ is determined by the dimensions of the atoms of B and by the action of R between these atoms.

We have discussed the Booleanization of a finite distributive lattice in Section 5. If we were to simply carry out that construction, we would need only choose relative complements

$$H_i \quad \text{to} \quad \hat{Z}_i \quad \text{in} \quad Z_i\,,$$
$$H'_j \quad \text{to} \quad \hat{Z}'_j \quad \text{in} \quad Z'_j\,,$$

and

$$H_{ij} \quad \text{to} \quad \hat{Z}_{ij} \quad \text{in} \quad Z_{ij}\,,$$

which would serve as the atoms of a Boolean algebra. (We would, of course, omit those which are zero.)

In our situation, however, we need to Booleanize the lattice, and, at the same time, analyze R. The action of R forces further non-lattice-theoretic relationships among the H_i, H'_j, and H_{ij}. Knowing these relationships in advance (Lemma 4), we will carefully select the spaces H_{ij} and we will decompose the spaces H_i and H'_j into spaces $\{G_{ik} \mid i \leqslant k \leqslant N\}$ and $\{G'_{jk} \mid j \leqslant k \leqslant N'\}$. The spaces H_{ij}, G_{ik}, and G'_{jk} will serve as the atoms of a Boolean algebra B which will include the lattice L; furthermore, because of our careful selection, the action of R between any two of the atoms will be essentially trivial.

LEMMA 4. (1) *If $i > 1$ and $j < N'$, then R induces an isomorphism R_{ij}: $Z_{ij}/\hat{Z}_{ij} \simeq Z_{i-1,j+1}/\hat{Z}_{i-1,j+1}$.*

(2) *If $i > 1$, then R induces a monomorphism R_i: $Z_i/\hat{Z}_i \to Z_{i-1}/\hat{Z}_{i-1}$.*

(3) *If $j \leqslant N'$, then R^{-1} induces a monomorphism R'_j: $Z'_{j+1}/\hat{Z}'_{j+1} \to Z'_j/\hat{Z}'_j$.*

Proof. We confine ourselves to the proof of clause (1). The other clauses are treated analogously.

It is immediate from the definitions that for any x in Z_{ij} there is a y in $Z_{i-1,j+1}$ such that xRy, and conversely for y in $Z_{i-1,j+1}$ there is an x in Z_{ij} so that xRy. To show that R induces an isomorphism R_{ij}: $Z_{ij}/\hat{Z}_{ij} \simeq Z_{i-1,j+1}/\hat{Z}_{i-1,j+1}$ we need only show that

$$(*) \qquad \hat{Z}_{ij}R \cap Z_{i-1,j+1} = \hat{Z}_{i-1,j+1}$$

and

$$(**) \qquad R\hat{Z}_{i-1,j+1} \cap Z_{ij} = \hat{Z}_{ij}\,.$$

By virtue of the symmetry inherent in the situation, we may confine ourselves to the treatment of $(*)$. Now since $\hat{Z}_{ij} = Z_{i-1,j} + Z_{i,j-1}$ and $\hat{Z}_{i-1,j+1} = Z_{i-2,j+1} + Z_{i-1,j}$, it is clear that $\hat{Z}_{i-1,j+1} \subseteq \hat{Z}_{ij}R$. Suppose conversely that $y \in \hat{Z}_{ij}R \cap Z_{i-1,j+1}$. Fix $x \in \hat{Z}_{ij}$ so that xRy. We may write more explicitly

$$x = x_1 + x_2\,, \qquad x_1 \in Z_{i-1,j}\,, \qquad x_2 \in Z_{i,j-1}\,.$$

This being so, we may find y_1, y_2 so that

$$x_1Ry_1, \qquad x_2Ry_2, \qquad y_1 \in Z_{i-2,j+1}, \qquad y_2 \in Z_{i-1,j}.$$

We know xRy and $xR(y_1 + y_2)$; we conclude that

$$0R(y - (y_1 + y_2)), \qquad \text{so} \quad y - (y_1 + y_2) \in Z'_1.$$

By assumption $y \in Z_{i-1,j+1}$, so certainly $y, y_1, y_2 \in Z_{i-1}$. Thus $y - (y_1 + y_2) \in Z_{i-1,1} \subseteq Z_{i-1,j} \subseteq \hat{Z}_{i-1,j+1}$; but of course $y_1 + y_2 \in \hat{Z}_{i-1,j+1}$, so finally $y \in \hat{Z}_{i-1,j+1}$. ∎

It is now an easy matter to choose the desired spaces and to describe the action of R on these spaces. We start with the spaces H_{ij}. For $i > N$, we have $Z_{ij} = \hat{Z}_{ij}$ and $H_{ij} = 0$. For $i \leqslant N$, let H_{i1} be any relative complement of $\hat{Z}_{i1}$ in Z_{i1} and use R to construct $H_{i-1,2}, H_{i-2,3}, \ldots, H_{1i}$ according to Lemma 4.1. In more detail, if a basis B_t^i for $H_{i-t,t+1}$ has been selected, then select $B_{t+1}^i \subseteq Z_{i-t-1,t+2}$ so that R induces a 1–1 correspondence $B_t^i \leftrightarrow B_{t+1}^i$, and so that the space $H_{i-t-1,t+2}$ spanned by B_{t+1}^i is a relative complement of $\hat{Z}_{i-t-1,t+2}$ in $Z_{i-t-1,t+2}$. Then R induces an isomorphism from each $H_{i-t,t+1}$ to $H_{i-t-1,t+2}$; R also takes each H_{1j} to 0 and takes 0 to each H_{i1}.

This situation can be represented pictorially as in Fig. 8. Here R maps each space isomorphically onto the one below it; each space on the bottom is mapped by R to 0 and each space on the top row is included in Z'_1.

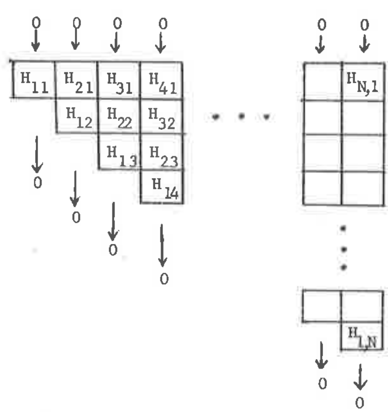


FIGURE 8

We next choose a relative complement H_N of $\hat{Z}_N$ in Z_N, with basis B_N. Then proceeding by downward induction on i, given a relative complement H_{i+1} of $\hat{Z}_{i+1}$ in Z_{i+1} with basis B_{i+1}, we construct a relative complement H_i of $\hat{Z}_i$ in Z_i with basis B_i as follows: By Lemma 4.2, we choose $A_i \subseteq Z_i$ so that

R induces a 1–1 correspondence $B_{i+1} \leftrightarrow A_i$, and then we extend A_i to a basis B_i for a complementary subspace H_i of $\hat{Z}_i$ in Z_i.

We now decompose $H_N, H_{N-1}, \ldots, H_1$ as follows. For each i, let $G_{i,i}$ be the subspace of H_i spanned by $B_i - A_i$. Then define $G_{i-1,i}, G_{i-2,i}, \ldots, G_{1,i}$ inductively by stipulating that $G_{k-1,i}$ is the image under R of $G_{k,i}$ in H_{k-1}.

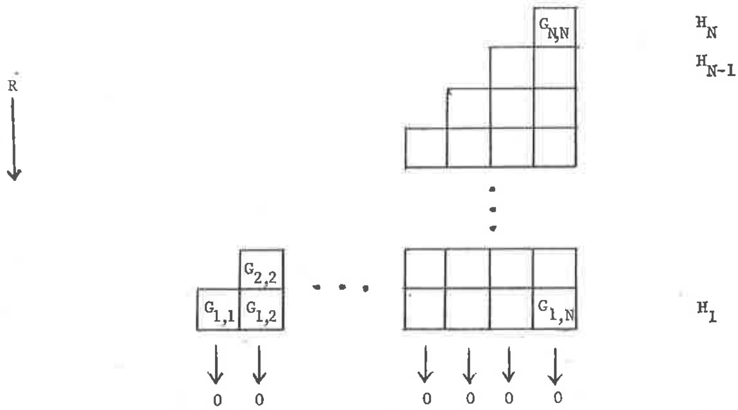


FIGURE 9

This situation can be represented pictorially as in Fig. 9. Here R maps each space isomorphically onto the one below it; each space on the bottom row is mapped to zero but the spaces on the top are not R-images at all. Note that the direct sum of the spaces in the ith row is H_i for each i, $1 \leqslant i \leqslant N$.

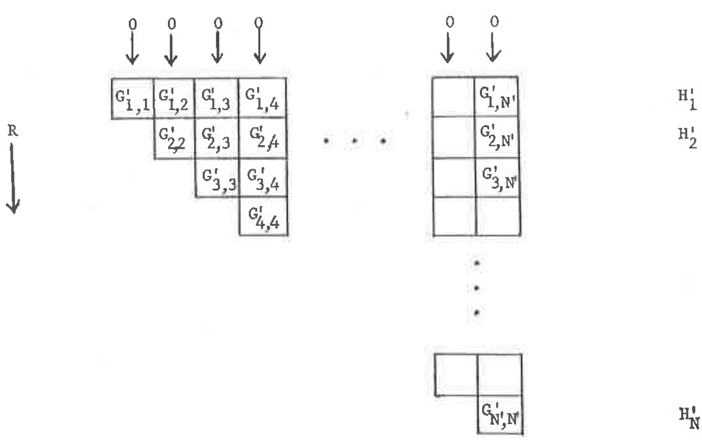


FIGURE 10

Similarly, using Lemma 4.3, we can define H'_j for $j = N', \ldots, 1$ and decompose each H'_j into $\{G'_{jk} \mid j \leqslant k \leqslant N'\}$, obtaining the pictorial representation shown in Fig. 10. Here R maps each space isomorphically onto the one below it; each space on the top row is included in Z'_1 but the spaces on the bottom are not in the domain of R at all. Note that the direct sum of the spaces in the jth row is H'_j.

At this point, we state the structure theorem that we have proved for nilpotent linear relations.

THEOREM 5. *Let* $\langle V; R\rangle$ *be a nilpotent linear relation, where* $Z = Z_N$, $Z' = Z'_{N'}$, *and* $N \leqslant N'$. *Then* V *can be written as a direct sum of spaces* $\{H_{ij} \mid 1 \leqslant i, j; i + j \leqslant N + 1\}$, $\{G_{ik} \mid i \leqslant k \leqslant N\}$, *and* $\{G'_{jk} \mid j \leqslant k \leqslant N'\}$ *so that*

(i) R *maps* H_{ij} *isomorphically onto* $H_{i-1,j+1}$ *for all* j *and all* $i > 1$;

(ii) R *maps* G_{ik} *isomorphically onto* $G_{i-1,k}$ *for all* $i > 1$ *and all* $k \geqslant i$;

(iii) R *maps* G_{jk} *isomorphically onto* $G'_{j+1,k}$ *for all* $j < k$;

(iv) $Z_i = \sum \oplus \{G_{tk} \mid t \leqslant i; t \leqslant k \leqslant N\} \oplus \sum \oplus \{H_{tj} \mid t \leqslant i; i + j \leqslant N + 1\}$;

(v) $Z'_j = \sum \oplus \{G'_{tk} \mid t \leqslant j; t \leqslant k \leqslant N'\} \oplus \sum \oplus \{H_{is} \mid s \leqslant j; i + j \leqslant N + 1\}$;

(vi) $Z_{ij} = \sum \oplus \{H_{ts} \mid t \leqslant i, s \leqslant j, i + j \leqslant N\}$;

(vii) R *is completely determined by* (i)–(v), *in the following precise sense:*

> *If* $x = \sum \{x_t \mid t \in T\}$ *is the decomposition of* x *and if, whenever either* $x_t \in H_{ij}$ *for some* $i > 1$ *or* $x_t \in G_{ik}$ *for some* $i > 1$ *or* $x_t \in G'_{jk}$ *for some* $j < N'$, *then* y_t *is defined to be the corresponding element of* $H_{i-1,j+1}$, $G_{i-1,k}$, *or* $G'_{j+1,k}$, *and otherwise* $y_t = 0$; *then* xRy *iff* $(y - \sum \{y_t \mid t \in T\}) \in Z'_1$.

Furthermore, the structure $\langle V; R\rangle$ *is completely determined by the dimensions of the spaces*

$$\{H_{i1} \mid i < N\} \cup \{G_{ii} \mid i \leqslant N\} \cup \{G'_{jj} \mid j \leqslant N'\}.$$

COROLLARY 6. *Let* $\langle V; R\rangle$ *be a nilpotent linear relation, where* V *is a vector space over a finite field. Then* $\langle V; R\rangle$ *is* $\aleph_0$*-categorical.*

Proof. Since each of the spaces Z_i, Z'_j, Z_{ij}, $\hat{Z}_i$, $\hat{Z}'_j$, and $\hat{Z}_{ij}$ is definable, we need only, by Theorem 5, write axioms which give the dimensions of $Z_N/\hat{Z}_N$, $Z'_{N'}/\hat{Z}'_{N'}$, and $Z_{1j}/\hat{Z}_{1j}$ for each $j < N$ over the given finite field, as well as the codimensions $[Z_{i-1}/\hat{Z}_{i-1}\colon R_i[Z_i/\hat{Z}_i]]$ and $[Z'_j/\hat{Z}'_j\colon R'_j[Z'_{j+1}/\hat{Z}'_{j+1}]]$. ∎

Corollary 7. *Let $\langle V; R\rangle$ be a nil linear relation; that is, $V = Z + Z'$. Then $\langle V; R\rangle$ is $\aleph_0$-categorical if and only if there is an n such that $Z = Z_n$ and $Z' = Z'_n$.*

7. Monomorphisms

In the preceding section we analyzed nilpotent linear relations. We now study the other extreme case of linear relations: monomorphisms. Thus let $\langle V; R\rangle$ be a linear relation where R is a monomorphism; by this we mean that R is a isomorphism from one subspace of V to another subspace of V. In keeping with our earlier notation, set

$$I = \{v \in V \mid \text{for all } n \in \mathbb{Z},\ R^n v \text{ is defined}\}.$$

Notice that Z and Z' are 0 and that R induces an automorphism $\bar{R}$ on I. In the special case where R is a monomorphism, Theorem 1 becomes

Theorem 8. *$\langle V; R\rangle$ is $\aleph_0$-categorical iff*

(i) *For some integer $n \geqslant 0$, $I = \{v \mid R^n v$ is defined$\}$, and*

(ii) *$\langle I; \bar{R}\rangle$ is $\aleph_0$-categorical.*

Proof. The necessity of conditions (i) and (ii) was verified in Section 4 in greater generality. (Note that since R is a monomorphism, condition 3 of Theorem 1 implies (i) here.) We deal here with the question of sufficiency.

For $0 \leqslant k, l$ define

$$V_{k,l} = \{v \in V \mid R^{-l}v \text{ and } R^k v \text{ are defined}\}.$$

If $k + l \geqslant n$, then $V_{k,l} = I$. We are interested in the lattice L of spaces generated by $\{V_{k,l} \mid k, l \geqslant 0\}$. Since $V_{k,l} = V_{k,0} \cap V_{0,l}$, and since $V_{k,0} \supseteq V_{k+1,0}$ and $V_{0,l} \supseteq V_{0,l+1}$, this lattice is generated by the chains $\{V_{k0} \mid k \geqslant 0\}$ and $\{V_{0l} \mid l \geqslant 0\}$. In particular it is finite and distributive, and the join-irreducible elements are among $\{V_{k,l} \mid 0 \leqslant k, l\}$. Define $\hat{V}_{k,l} = V_{k+1,l} + V_{k,l+1}$. In the free modular lattice on $\{V_{k,0}\}$ and $\{V_{0,l}\}$, $V_{k,l}$ covers $\hat{V}_{k,l}$. In L, $V_{k,l}$ covers or equals $\hat{V}_{k,l}$. We will Booleanize L by choosing relative complements $K_{k,l}$ of $\hat{V}_{k,l}$ in $V_{k,l}$.

As in Section 4, R induces isomorphisms $R_{k,l}$: $V_{k,l}/\hat{V}_{k,l} \simeq V_{k-1,l+1}/\hat{V}_{k-1,l+1}$ for $l \geqslant 1$. Thus if we let K_k be a relative complement of $\hat{V}_{k,0}$ in $V_{k,0}$, then for each $l \leqslant k$ we may take $R^l K_k$ as a relative complement $K_{l,k-l}$ of $\hat{V}_{l,k-l}$ in $V_{l,k-l}$.

We thus obtain the pictorial representation shown in Fig. 11. Here R maps each space isomorphically onto the one below it.

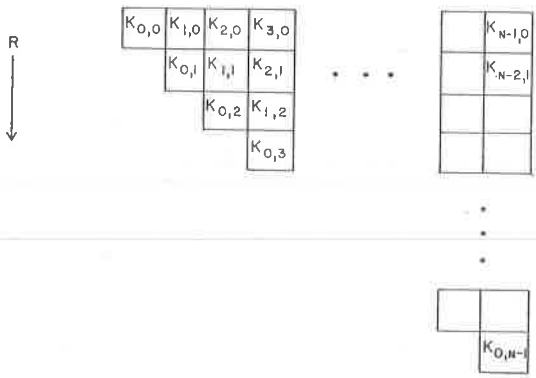


FIGURE 11

We can think of this structure in the following way: Since R is a monomorphism, every element x of V is a member of a unique maximal R-chain of elements of V; this R-chain either has the form $x_1Rx_2Rx_3 \cdots x_kRx_{k+1}$ for some k, $0 \leqslant k \leqslant N$, or else it is a two-way infinite chain. In particular, if $x = x_1 \in K_{k,0}$, then the maximal chain is of length $k+1$ and $x_1 \in K_{k,0}$, $x_2 \in K_{k-1,1}, \ldots, x_{k+1} \in K_{0,k}$. Since $K_{0,0}$ represents the part of V which is unaffected by R, the only part of V missing from Fig. 11 is I.

Thus the linear relation $\langle V; R\rangle$ is completely determined by the dimensions of the spaces $\{K_{k,0} \mid 0 \leqslant k \leqslant N-1\}$ and the definable substructure $\langle I; \bar{R}\rangle$. Hence, since we are assuming that $\langle I; \bar{R}\rangle$ is $\aleph_0$-categorical, the same is true for $\langle V; R\rangle$. ▮

To complete this section, we present a criterion for the $\aleph_0$-categoricity of automorphisms.

THEOREM 9. *Let I be a vector space and let R be an automorphism of I. Then the following are equivalent:*

(1) *$\langle I; R\rangle$ is $\aleph_0$-categorical.*

(2) *There is an integer $m \geqslant 0$ such that given $x \in I$ there is an $s \leqslant m$ and elements $x_1, x_2, \ldots, x_s$ of I such that*

$$xRx_1Rx_2 \cdots x_{s-1}Rx_sRx.$$

(3) *There is a polynomial $p(t)$ in $F[t]$ such that $p(R) = 0$ as an endomorphism of I.*

Proof. That (1) implies (2) is a consequence of the Basic Theorem on $\aleph_0$-categoricity. If (2) holds, then $p(R) = 0$ where $p(t) = \prod_{n \leqslant m} t^n - 1$, so that (3) holds.

Finally, assume that (3) holds. We view I as an $F[t]$-module in the usual fashion, namely, by setting $t \cdot x = Rx$ for $x \in I$. Then I is a module over the principal ideal domain $F[t]$ and $p(t)$ annihilates I. Under these circumstances it is well-known, and easily proved, that I is a direct sum of cyclic $F[t]/(p(t))$-modules. (This is analogous to the case of Abelian torsion groups of bounded exponent.) Since there are only finitely many distinct cyclic $F[t]/(p(t))$-modules, it follows that $\langle I; R\rangle$ is $\aleph_0$-categorical. We refer the reader to Kaplansky [8], where such situations are discussed algebraically, or Eklof and Fisher [6], where the case of Abelian torsion groups is discussed, or Baur [2], where this is a consequence of the general result. ∎

COROLLARY 10. *The monomorphism $\langle V; R\rangle$ is $\aleph_0$-categorical iff*

(i) *For some integer $n \geqslant 0$, $I = \{v \mid R^n v$ is defined$\}$.*

(ii) *For some integer $m \geqslant 0$, given any $x \in I$ there is an $s \leqslant m$ and elements $x_1, x_2, \ldots, x_s$ of I such that $xRx_1Rx_2 \cdots x_{s-1}Rx_sRx$.*

8. SUFFICIENCY OF CONDITIONS 1–3 FOR $\aleph_0$-CATEGORICITY

Let $\langle V; R\rangle$ be a linear relation. Let $Z = \bigcup R^k 0$ and let $Z' = \bigcup 0R^l$ be as before, and let I be as before. In addition, let $D = \{v \mid$ for some v', $vRv'\}$ be the domain of R and let $D' = \{v \mid$ for some v', $v'Rv\}$ be the range of R.

In this section, we will put together the results of Sections 6 and 7 to obtain a proof of Theorem 1. That they can be put together is a consequence of the following lemma.

LEMMA 11. *Assume that $Z = Z_N$ and $Z' = Z'_{N'}$. Then the lattice L of subspaces of V generated by $\{D, D', I, Z, Z'\}$ is finite and distributive.*

Proof. Consider the modular lattice L' on the generators $\{d, d', i, z, z'\}$ subject only to the relations

$$z \leqslant d, \qquad z' \leqslant d', \qquad i \leqslant d \cap d', \qquad i \cap (z + z') = z \cap z'.$$

We claim that L is a homomorphic image of L' and that L' is finite and distributive.

Now L is a homomorphic image of L' since we have proved that all of the relations above are true in L—the last having been derived in Section 4 from the assumption that $Z = Z_N$ and $Z' = Z'_{N'}$.

Turning now to the structure of L', we recall that the structure of the lattice L_0 generated by the two chains $\{z, d\}$ and $\{z', d'\}$ is as shown in Fig. 12. To this lattice we must add an element i which is below $d \cap d'$ and whose intersection with $z + z'$ is $z \cap z'$. Having added i, we must also add $a + i$ for each

$a \in L_0$. But for $a \geqslant d \cap d'$, $a + i$ must be a; so we need only add $a + i$ for $z \cap z' \leqslant a \leqslant z + z'$. We claim that the resulting set K is closed under $\cap$, and therefore it is identical with L'.

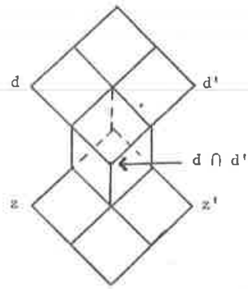


FIGURE 12

Thus we need to verify that

(i) if $a, b \leqslant z + z'$, then $a \cap (b + i) \in K$;

(ii) if $a \geqslant d \cap d'$, $b \leqslant z + z'$, then $a \cap (b + i) \in K$;

(iii) if $a, b \leqslant z + z'$, then $(a + i) \cap (b + i) \in K$.

As to (i):

$$\begin{aligned} a \cap (b + i) &= (a \cap (z + z')) \cap (b + i) \\ &= a \cap [(z + z') \cap (b + i)] \\ &= a \cap [b + ((z + z') \cap i)] \qquad \text{(by modularity)} \\ &= a \cap [b + (z \cap z')] = a \cap b. \end{aligned}$$

As to (ii): $a \cap (b + i) = (a \cap b) + i$, by modularity, since $i \leqslant d \cap d' \leqslant a$.
As to (iii):

$$\begin{aligned} (a + i) \cap (b + i) &= (a \cap (b + i)) + i \qquad \text{(by modularity)} \\ &= (a \cap b) + i \qquad \text{by (i).} \end{aligned}$$

Hence K is identical with L'. It is easily seen that L' has the representation shown in Fig. 13. The verification that L' is distributive is direct. Hence its homomorphic image L, the lattice of subspaces of V generated by $\{D, D', I, Z, Z'\}$ is finite and distributive. ∎

We will now Booleanize the lattice L which is a homomorphic image of the lattice L' pictured above. We first Booleanize $Z + Z'$ as in Section 6; that is, we restrict our attention to the nilpotent linear relation $\langle Z + Z'; R_N \rangle$ (where R_N is R restricted to $Z + Z'$) and imbed the lattice generated by the two chains of subspaces $\{Z_i \mid i \leqslant N\}$ and $\{Z'_j \mid j \leqslant N'\}$ into a Boolean algebra of subspaces of $Z + Z'$, taking care that the action of R between any two atoms is essentially trivial. By referring to Figs. 8–10 in the proof of Theorem 5, one can see that the spaces $D \cap Z'$ and $D' \cap Z$ are in this Boolean algebra and therefore the same holds true of the nine possible subspaces of $Z + Z'$ in the lattice L.

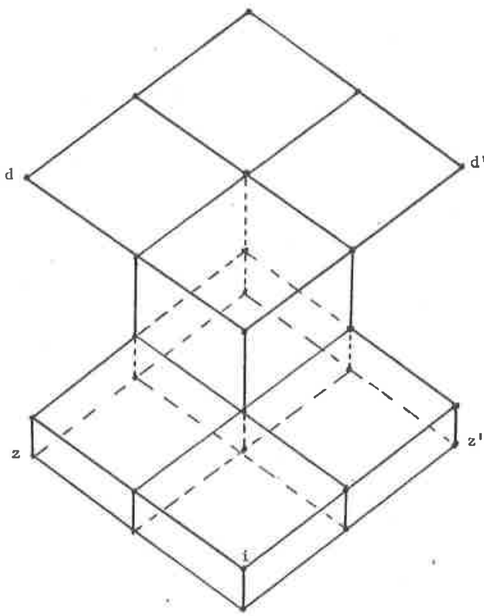


FIGURE 13

We next choose a complement of $Z \cap Z'$ in I. Since we are assuming that the linear relation $\bar{R}$ induced on $I/Z \cap Z'$ is an $\aleph_0$-categorical automorphism, we can apply the analysis of Theorem 9. That is, there is an m such that $I/Z \cap Z' = \oplus\sum_{s\leqslant m} I_s$ where each I_s is a direct sum $\oplus\sum_{t=0}^{K(s)} I_{s,t}$ of isomorphic cyclic $F(t)$-modules. (Note that $K(s)$ may be infinite.) Furthermore, $I_{s,t}$ has a basis $[d_1],..., [d_q]$ such that

$$(*) \qquad [d_1]\, \bar{R}[d_2] \cdots [d_{q-1}]\, \bar{R}[d_q]\, \bar{R}[d_{q+1}],$$

where $[d]$ denotes $d + (Z \cap Z')$, and where $[d_{q+1}]$ is a linear combination $\sum_{i\leqslant q} \alpha_i[d_i]$, with each $\alpha_i \in F$.

For each $s \leqslant m$ and each t, $0 \leqslant t \leqslant K(s)$, we will define a subspace $C_{s,t}$ so that $I = (Z \cap Z') \oplus\sum_{s\leqslant m} C_s$ where each $C_s = \oplus\sum_{t=0}^{K(s)} C_{s,t}$, and such that each $C_{s,t}$ is invariant under R—that is, for each $v \in C_{s,t}$ there is a unique $v' \in C_{s,t}$ with vRv'. Furthermore we will have $\langle C_{s,t} ; R\rangle \simeq \langle I_{s,t} ; \bar{R}\rangle$.

Indeed, given $d_1, d_2, \ldots, d_{q+1}$ as in (*) above, we can choose $c_1, c_2, \ldots, c_{q+1}$ so that $c_i \in [d_i]$ for each i and $c_1 R c_2 R c_3 \cdots c_q R c_{q+1}$. In particular, $c_{q+1} = \sum_{i \leqslant q} \alpha_i c_i + z$ for some $z \in Z \cap Z'$. Suppose more particularly that $z \in Z \cap Z'_J$ with $1 \leqslant J \leqslant N'$, and that $c_1, c_2, \ldots, c_{q+1}$ have been chosen so as to minimize J. Fix $z_1, z_2, \ldots, z_{q+1}$ with $z_{q+1} = z$, $z_1 R z_2 R \cdots R z_q R z_{q+1}$, and $z_i \in Z'_{J+i-(q+1)}$ for each i for which $J + i - (q+1) > 0$ and $z_i = 0$ for each i for which $J + i - (q+1) \leqslant 0$. Let $e_i = c_i - z_i$ for each i. Then $e_1 R e_2 \cdots R e_q R e_{q+1}$, each $[e_i] = [d_i]$, and $e_{q+1} = \sum \alpha_i e_i + z'$, where $z' = \sum \alpha_i z_i \in Z_{J-1}$ if $J > 1$ and $z' = 0$ if $J = 1$. Hence $J = 1$ and $e_{q+1} = \sum \alpha_i e_i$. We now define $C_{s,t} = \langle e_1, e_2, \ldots, e_s \rangle$ and note that it is invariant under R as specified. (Note that if $x \in C_{s,t}$ and $y \in C_{s,t}$ and xRy, then xRz iff $0R(y - z)$ iff $y - z \in Z'_1$; thus the relationships between elements of $C_{s,t}$ and other elements of V are completely specified since each $C_{s,t}$ is invariant under R and since Z'_1 is specified.)

The remaining part of the lattice L' is represented in Fig. 14, where $N = (I + Z + Z') \cap (D \cap D')$.

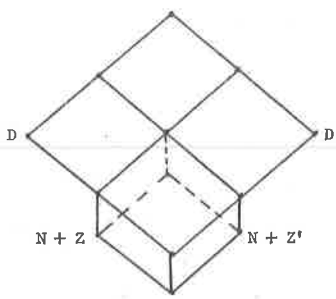


FIGURE 14

We note that if cRd and either c or d is in $Z + Z'$, then so is the other, and thus the only R-relations involving elements of $Z + Z'$ are with other elements of $Z + Z'$. For example, if $c \in Z + Z'$ and cRd, we write $c = a + b$ where $a \in Z$ and $b \in Z'$. Since $a \in Z$, aRa' for $a' \in Z$. Hence $bR(d - a')$. Since $b \in Z'$, $d - a' \in Z'$. But then $d = a' + (d - a') \in Z + Z'$.

Similarly, if cRd and either c or d is in $I + (Z + Z')$, then so is the other. For example, if $c \in I + (Z + Z')$ and cRd, then we write $c = a + b$ where $a \in I$ and $b \in Z + Z'$. But aRa' for $a' \in I$ so $bR(d - a')$. By the above, since $b \in Z + Z'$, also $d - a' \in Z + Z'$. Hence $d = a' + (d - a') \in I + (Z + Z')$. Moreover, we see that if cRd where $c, d \in I + (Z + Z')$, then we can write $c = c_1 + c_2 + c_3$ and $d = d_1 + d_2 + d_3$ where $c_1, d_1 \in I$, $c_2, d_2 \in Z$, $c_3, d_3 \in Z'$ and $c_1 R d_1$, $c_2 R d_2$, $c_3 R d_3$.

Thus the only R-relations involving elements of $I + (Z + Z')$ are with other elements of $I + (Z + Z')$; moreover, any such R-relation is a consequence of R-relations between elements of I, between elements of Z, and between elements of Z'.

We let P denote $I + (Z + Z')$. Let $W = V/P$ and let $\bar{R}$ be the linear relation induced on V/P by R—thus $[c]\, \bar{R}[d]$ if for some $u \in [c]$ and some $v \in [d]$ we have uRv. The domain of $\bar{R}$ is $\{[x] \mid x \in D\}$ and the range of $\bar{R}$ is $\{[y] \mid y \in D'\}$. Furthermore, if $[x]\, \bar{R}[y_1]$ and $[x]\, \bar{R}[y_2]$, then $[0]\, \bar{R}[y]$ where $y = y_1 - y_2$. We claim then that $y \in P$ so that $\bar{R}$ is single-valued on its domain. Indeed, if $dR(y + d')$ where $d, d' \in P$ then, by the remarks above, $y + d' \in P$ and hence $y \in P$. Similarly $\bar{R}$ is 1–1 on its domain, so that $\langle W; R \rangle$ is a monomorphism.

We may thus apply our analysis of monomorphisms in Section 7. Suppose first that

$$[c_1]\, \bar{R}[c_2] \cdots [c_m]\, \bar{R}[c_{m+1}].$$

We claim that there are elements $d_1, d_2, \ldots, d_m, d_{m+1}$ such that $[d_i] = [c_i]$ for each i and such that $d_1 R d_2 \cdots d_m R d_{m+1}$. To verify this, we show by induction on s, $1 \leqslant s \leqslant m + 1$, that there is a sequence $d_1{}^s, d_2{}^s, \ldots, d_s{}^s$ such that $[d_i{}^s] = [c_i]$ for all $i \leqslant s$ and such that $d_1{}^s R d_2{}^s \cdots d_{s-1}^s R d_s{}^s$. For the basis step $s = i$, we can take $d_1{}^1 = c_1$. For the induction step, assume that $d_1{}^s, d_2{}^s, \ldots, d_s{}^s$ are as required. Since $[d_s{}^s]\, \bar{R}[c_{s+1}]$, we get $(d_s{}^s + p)\, R(c_{s+1} + p')$ where $p, p' \in P = (I + Z) + Z'$. Set $p = p\ + p_2$ where $p\ \in I + Z$, $p_2 \in Z'$; now $p_1 R q$ for some $q \in I + Z$, so that $(d_s{}^s + p_2)\, R(c_{s+1} + p'_2)$ where $p_2 \in Z'$ and $p'_2 \in P$. Since $p_2 \in Z'$, there are $q_1, q_2, \ldots, q_{s-1}$ all in Z' such that $q_1 R q_2 \cdots q_{s-1} R p_2$. Set $d_i^{s+1} = d_i{}^s + q_i$ for $i \leqslant s - 1$, $d_{s+1}^s = d_s{}^s + p_2$, $d_{s+1}^{s+1} = c_{s+1} + p'_2$. Then the induction hypothesis continues to hold, so we arrive at $d_1, d_2, \ldots, d_m, d_{m+1}$ with the desired properties.

It now follows that if

$$[c_1]\, \bar{R}[c_2] \cdots [c_m]\, \bar{R}[c_{m+1}], \qquad \text{then} \quad m \leqslant 2n.$$

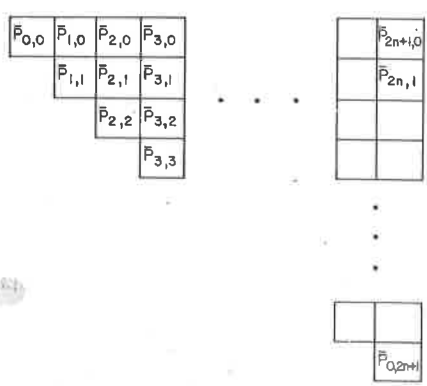


FIGURE 15

(Recall that condition 3 of Theorem 1 states that given $\{v_i \mid -n \leqslant i \leqslant n\}$ with $v_i R v_{i+1}$ for $-n \leqslant i < n$, then $v_0 \in I$.) Thus the invariant subspace corresponding to I in the monomlrphisms considered in Section 7 is here (0). Hence $\langle W; \bar{R}\rangle$ can be represented by Fig. 15. Here R maps each space $\bar{P}_{s,j} = P_{s,j}/P$ isomorphically onto the space below it.

We carefully select for each $s \leqslant 2n + 1$ and each $j \leqslant s$ a complement $K_{s,j}$ of P in $P_{s,j}$. Indeed, given a basis $\{[c^\alpha] \mid \alpha \in A_s\}$ of $P_{s,0}/P$, we find for each $\alpha \in A$ a sequence

$$[c^\alpha] = [c_1^\alpha]\ \bar{R}[c_2^\alpha] \cdots [c_{s-1}^\alpha]\ \bar{R}[c_s^\alpha]$$

with each $[c_j^\alpha] \in P_{s,j}/P$ and then find elements $d_1^\alpha, d_2^\alpha, \ldots, d_s^\alpha$ so that each $[d_j^\alpha] = [c_j^\alpha]$ and so that $d_1^\alpha R d_2^\alpha \cdots d_s^\alpha$. Clearly $K_{s,0} = \langle d_1^\alpha \mid \alpha \in A_s\rangle$ is a complement of P in $P_{s,0}$; furthermore $K_{s,j} = \langle d_j^\alpha \mid \alpha \in A_s\rangle$ is a complement of P in $P_{s,j}$. This completes the Booleanization since we now have a complement of N in $D \cap D'$, namely, $\sum \oplus \{K_{s,j} \mid 0 < j < s\}$, a complement of $(D \cap D') + (N + Z')$ in D', namely, $\sum \oplus \{K_{s,s} \mid 1 \leqslant s \leqslant n\}$, a complement of $(D \cap D') + (N + Z)$ in D, namely, $\sum \oplus \{K_{s,0} \mid 1 \leqslant s \leqslant n\}$, and a complement of $D + D'$ in V, namely, $K_{0,0}$.

Summarizing the proof of Theorem 1, we see that if $\langle V; R\rangle$ satisfies 1–3, then the structure of $\langle V; R\rangle$ is completely determined by the following invariants:

(1) For each $i \leqslant N$, the dimension of $Z_{i,1}/Z_{i-1,1}$.

(2) The dimension of $Z_N/\hat{Z}_N$.

(3) For each i, $1 < i < N$, the dimension of the complement of the image of $Z_i/\hat{Z}_i$ in $Z_{i-1}/\hat{Z}_{i-1}$ under R.

(4) The dimension of $Z'_{N'}/\hat{Z}'_{N'}$.

(5) For each j, $1 < j < N'$, the dimension of the complement of the image of $Z'_j/\hat{Z}'_j$ in $Z'_{j-1}/\hat{Z}'_{j-1}$ under R^{-1}.

(6) For each $s \leqslant 2m$, the number $K(s) \leqslant \aleph_0$ of isomorphic $C_{s,t}$'s which comprise C_s.

(7) The dimensions of the spaces $\{K_{s,0} \mid 0 \leqslant s \leqslant 2n + 1\}$.

Thus $\langle V; R\rangle$ is completely determined by specifying these invariants and the action of R as described in this section. Hence $\langle V; R\rangle$ is $\aleph_0$-categorical. ∎

9. $\aleph_0$-Categorical Groups and Modules

In the preceding sections, we have obtained an explicit algebraic criterion for the $\aleph_0$-categoricity of linear relations $\langle V; R\rangle$. Recall that with each group $G = \langle A, d, e\rangle$, where A has exponent 2 and $G/A \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, we associated a linear relation $\langle A; R\rangle$ via the $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$-module structure of A. We will now show that the linear relation $\langle A; R\rangle$ is $\aleph_0$-categorical if and only if A is $\aleph_0$-categorical as a $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$-module. By the general theorem connecting $\aleph_0$-categoricity of groups with $\aleph_0$-categoricity of the associated modules (Theorem 17), we can deduce that G is $\aleph_0$-categorical if and only if $\langle A; R\rangle$ is $\aleph_0$-categorical.

Recall that in preceding sections A is viewed as an $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$-module where $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$ is the group ring generated over F_2 by the two linear transformations δ and ϵ defined by

$$\delta(a) = d^{-1}ad \qquad \text{and} \qquad \epsilon(a) = e^{-1}ae.$$

We can also regard $F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$ as generated by $x = 1 + \delta$ and $y = 1 + \epsilon$. Note that $x^2 = y^2 = 1$ and $xy = yx$, so that $S = F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$ is also $F_2[x, y]/(x^2, y^2)$ and an S-module is just an F_2-vector space equipped with two commuting nilpotent transformations of order 2.

We passed from the S-module A to the linear relation R by specifying that

$$\langle v, w\rangle \in R \qquad \text{iff for some } a \in A,\ v = xa \text{ and } w = ya.$$

Thus, informally, the action of R can be thought of as yx^{-1}, so that in passing from the S-module A to the linear relation $\langle A; R\rangle$ we have lost the actions of x and y although we have added the "product" action yx^{-1}. We will now see that what has been lost does not seriously affect the analysis of the preceding sections and indeed does not affect $\aleph_0$-categoricity at all.

THEOREM 12. *A is an $\aleph_0$-categorical S-module if and only if $\langle A; R\rangle$ is an $\aleph_0$-categorical linear relation.*

Remark. The following fairly lengthy argument is illustrated by Fig. 16 (extending Fig. 13), using a notation we will now describe.

Notation 13. 1. $K_x = \ker x$, $K_y = \ker y$, $K_{xy} = \ker xy$;

2. $A^x = \operatorname{Im} x$, $A^y = \operatorname{Im} y$, $A^{xy} = \operatorname{Im} xy$;

3. $K_x{}^y = y[K_x]$, $K_y{}^x = x[K_y]$, $K^x_{xy} = x[K_{xy}]$, $K^y_{xy} = y[K_{xy}]$

With $R = yx^{-1}$ in the sense defined above, we will find that several of these subspaces in fact coincide with spaces introduced previously in the analysis of the general $\aleph_0$-categorical linear relation R; the others are accounted for if that analysis is performed with a little extra care. For this reason, Theorem 12 may be proved roughly as follows:

Step 1: Carry out the Booleanization of the lattice of spaces associated with R.

Step 2: Describe the action of x and y on the atoms of the resulting Boolean algebra.

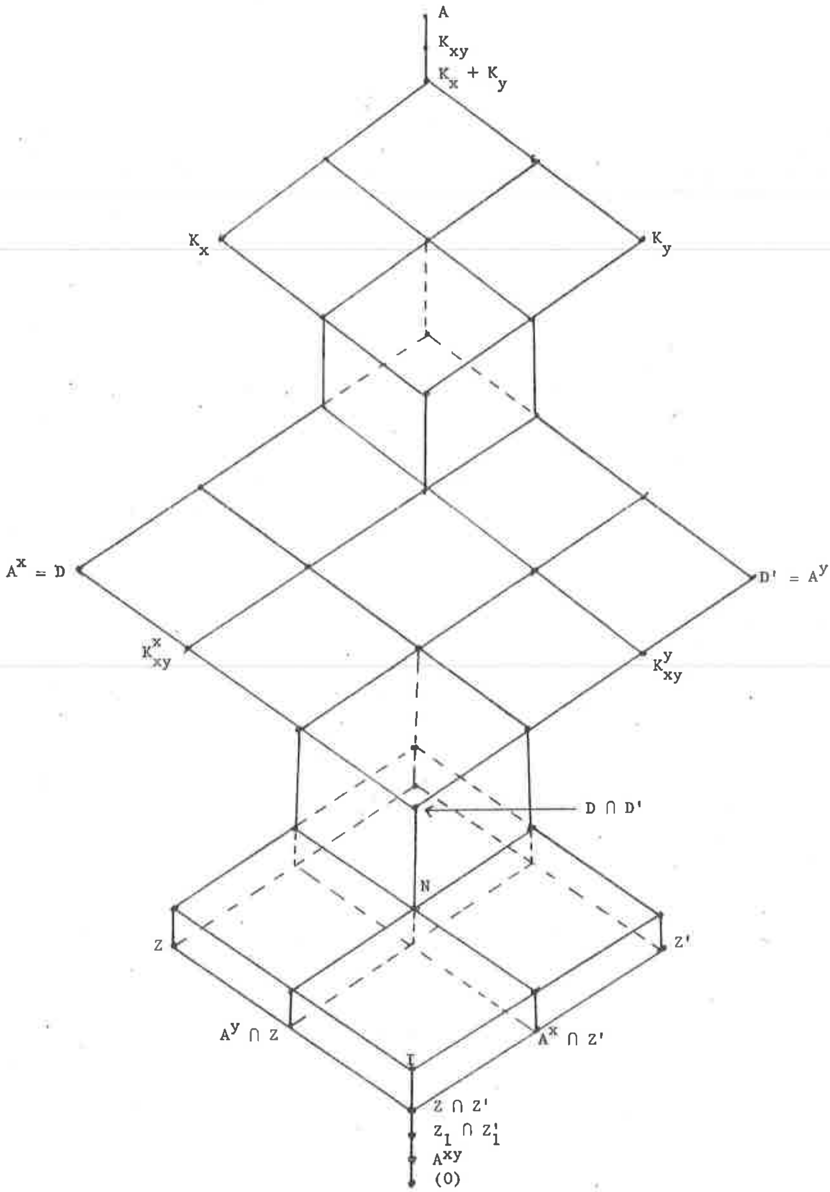


FIGURE 16

In fact a rather specific Booleanization must be chosen in Step 1 in order that the description sought in Step 2 will be available.

Throughout the rest of this section we let L be the lattice of subspaces generated by all of the following:

1. the chain $K_{xy} \supseteq K_x \supseteq A^x \supseteq Z \supseteq A^{xv}$;
2. the chain $K_{xy} \supseteq K_y \supseteq A^v \supseteq Z' \supseteq A^{xv}$;
3. the subspace I.

After seeing that L is a finite distributive lattice we will Booleanize it with care, as needed for Step 2 above. A further prerequisite is

Step 2a: Booleanize L so that the additional spaces $K_x{}^v$, $K_y{}^x$, K_{xy}^x, and K_{xy}^v are in the resulting Boolean algebra.

The first order of business is to determine the structure of the lattice L.

LEMMA 14.

1. $D = A^x, \qquad D' = A^v$;
2. $K_y{}^x = Z_1, \qquad K_x{}^y = Z_1'$;
3. $A^x \cap K_y = K_{xy}^x, \qquad K_x \cap A^v = K_{xy}^v$;
4. $I + Z + Z' \subseteq K_x \cap K_y$.

Proof. We need prove only 4; the remaining parts are trivial. Note first that for $u, v, w \in A$ the relation $uRvRw$ entails $u \in K_y$. Indeed, since we can solve the equations

$$xa = u, \quad ya = v; \qquad xb = v, \quad yb = w$$

with $a, b \in A$, we find $yu = xya = xv = x^2b = 0$. Thus it follows that $I + Z \subseteq K_y$. Furthermore $Z' \subseteq D' \subseteq K_y$, so $I + Z + Z' \subseteq K_y$. Similarly $I + Z + Z' \subseteq K_x$, proving 4. ∎

The structure of L now becomes transparent. That part of L which is generated by the two chains Z, A^x, K_x and Z', A^v, K_y is a homomorphic image of the free modular lattice $F(3, 3)$ modulo the relation $(c_1 + c'_1) \leqslant (c_3 \cap c'_3)$, which is true in L by Lemma 14.4. The addition of I is as in Section 8, Fig. 13. We then need only add the spaces $(0) \subseteq A^{xv} \subseteq Z_1 \cap Z'_1 \subseteq Z \cap Z'$ at the bottom, and the spaces $K_x + K_y \subseteq K_{xv} \subseteq A$ at the top. Thus L is a homomorphic image of the lattice in Fig. 16.

In the course of the proof of Theorem 12 (below) step 2a will present only minor difficulties. The most troublesome spaces in this connection are K_{xy}^x and K_{xy}^v. It is convenient to insert a preliminary observation couched in the terminology of Section 8, particularly Figs. 13 and 15.

LEMMA 15. *Set*

$$B = ((D \cap D') + Z + \bigoplus\sum \{K_{s,0} \mid 2 \leqslant s \leqslant 2n + 1\}),$$
$$B' = ((D \cap D') + Z' + \bigoplus\sum \{K_{s,s} \mid 2 \leqslant s \leqslant 2n + 1\}).$$

Then $B \subseteq K_{xy}^x$ *and* $B' \subseteq K_{xv}^v$.

Proof. Trivial.

Referring to Figs. 15 and 16 and the discussion in Section 8, we recall that $\oplus\sum\{K_{s,0} \mid 1 \leqslant s \leqslant 2n+1\}$ is a complementary subspace to $(D \cap D') + Z$ in D. We have now placed K^x_{xy} between $(D \cap D') + Z$ and D. Lemma 15 says that all but $K_{1,0}$ of the sum above is actually in K^y_{xy}.

Proof of Theorem 12. We begin by embedding the lattice of subspaces of A generated by $\{Z, Z', D, D', I\}$ in a Boolean algebra $\mathscr{B}$ as prescribed in Sections 6–8. Extend $\mathscr{B}$ to an algebra $\mathscr{B}_1$ by adjoining as new atoms the space A^{xy} and a complement C_1 to A^{xy} in Z_{11}.

Because x and y act trivially on $K_x \cap K_y$, that part of $\mathscr{B}_1$ lying below $K_x \cap K_y$ need not be altered except as required by step 2a (cf. Fig. 16). But by Lemma 15 we may account for the spaces K^x_{xy} and K^y_{xy} as follows

> Choose a complement $K'_{1,0}$ to B in K^x_{xy} and choose a complement $K''_{1,0}$ to K^x_{xy} in A^x. Redefine $K_{1,0}$ by setting $K_{1,0} = K'_{1,0} \oplus K''_{1,0}$.

Notice that this procedure implies a corresponding redefinition of $K_{0,1}$ since, in Section 8, $K_{0,1}$ is selected so that R induces an isomorphism $K_{1,0} \simeq K_{0,1}$. In particular the decomposition of $K_{1,0}$ above induces a decomposition

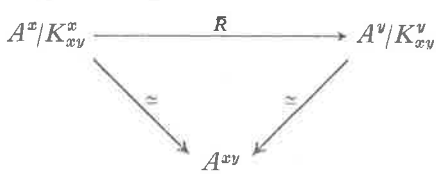
$$K_{0,1} = K'_{0,1} \oplus K''_{0,1}$$

and since we have

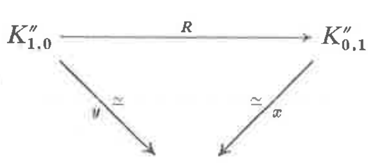
$$BR = B', \qquad K^x_{xy}R = K^y_{xy}, \qquad A^xR = A^y,$$

it follows that $K'_{0,1}$ is a complement to B' in K^y_{xy} and $K''_{0,1}$ is a complement to K^y_{xy} in A^y.

Call the extension of $\mathscr{B}_1$ defined in this way $\mathscr{B}_2$. Notice that the commutative diagram



(diagonally vertical isomorphisms are induced by y and x) yields a commutative diagram



This means that the action of x and y on the atoms of $\mathscr{B}_2$ is known in a coherent way.

At this stage the complement of $D + D'$ in $\mathscr{B}_2$ is a single atom denoted $K_{0,0}$ in Section 8. We may assume $K_{0,0} = C_2 \oplus C'_2$ where C_2 is a complement to $K^x_{xy} + K^y_{xy}$ in $K_x \cap K_y$ and C'_2 is a complement to $A^x + A^y + (K_x \cap K_y)$ in A. Calling the resulting Boolean algebra of spaces $\mathscr{B}_3$, we will only find it necessary to modify the choice of C'_2.

C'_2 will be the direct sum of certain spaces:

C_3, a complement to $(K_x \cap K_y) + A^x$ in K_x;
C_4, a complement to $(K_x \cap K_y) + A^y$ in K_y;
C_5, a complement to $K_x + K_y$ in K_{xy};
C_6, a complement to K_{xy} in A.

The selection of C_3 and C_4 is carried out bearing in mind the isomorphisms $\bar{y}$, $\bar{x}$ induced by y, x as follows:

$$\bar{y}\colon K_x/(A^x + (K_x \cap K_y)) \simeq K_x{}^y/A^{xy} = Z'_1/A^{xy},$$
$$\bar{x}\colon K_y/(A^y + (K_x \cap K_y)) \simeq K_y{}^x/A^{xy} = Z_1/A^{xy}.$$

Now Z'_1 is a direct sum of various spaces (see Figs. 8 and 10) including Z_{11}, which has been decomposed into $A^{xy} \oplus C_1$. We select for each of the components of Z'_1, except for A^{xy}, a subspace of K_x which y maps isomorphically onto the given component; and then we define C_3 to be the sum of these subspaces of K_x so that C_3 is a sum of atoms and is a complementary subspace to $A^x + (K_x \cap K_y)$ in K_x. Similarly we select a complementary subspace C_4 to $A^y + (K_x \cap K_y)$ in K_y.

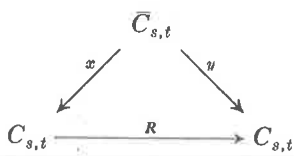
We next construct the relative complement C_5 of $K_x + K_y$ in K_{xy}. As we observed much earlier, y maps K_{xy} to $K_x \cap D' = K^y_{xy}$. It is easily verified that modulo $K_x + K_y$ this map is 1–1. Similarly, modulo $K_x + K_y$, x maps K_{xy} isomorphically onto $K_y \cap D = K^x_{xy}$. Thus we must find for each space in the Booleanization of K^x_{xy} a space in K_{xy} which x maps to it, and we must find for each space in the Booleanization of K^y_{xy} a space in K_{xy} which y maps to it. Moreover, this must be done coherently.

Thus, for example, within $Z \cap Z'$ the spaces $\{H_{ij} \mid 2 \leqslant i + j \leqslant N + 1$ and $i \geqslant 2\}$ above the bottom diagonal in Fig. 8 are not yet x-images of spaces already selected, although those on the bottom diagonal are already x-images. Similarly, the spaces $\{H_{ij} \mid 2 \leqslant i + j \leqslant N + 1$ and $j \geqslant 2\}$ below the top row in that diagram are not yet y-images, although those on the top row are y-images. For each H_{ij} where $i \geqslant 2$ we select a subspace $\bar{H}_{ij}$ of K_{xy} so that x maps $\bar{H}_{ij}$ isomorphically onto H_{ij} and y maps $\bar{H}_{ij}$ isomorphically onto $H_{i-1,j+1}$. This is done using the fact that R maps H_{ij} isomorphically on $H_{i-1,j+1}$; indeed, if we choose a basis $\{v_i\}$ for H_{ij} and a basis $\{w_i\}$ for $H_{i-1,j+1}$

so that $v_t R w_t$ for all t, we can let $\bar{H}_{ij}$ be spanned by $\{a_t\}$ where for each t, $v_t = xa_t$ and $w_t = ya_t$.

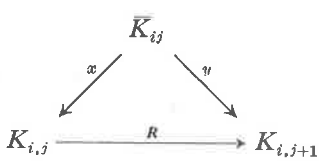
The same procedure is employed for the spaces $\{G_{i,j}\}$ which complement $Z \cap Z'$ in Z, and for the spaces $\{G'_{i,j}\}$ which complement $Z \cap Z'$ in Z'. Thus for each $G_{i,j}$ in Fig. 9 with $i > 1$ we use the fact that R maps $G_{i,j}$ isomorphically onto $G_{i-1,j}$ to select a space $\bar{G}_{i,j}$ such that x maps $\bar{G}_{i,j}$ isomorphically onto $G_{i,j}$ and y maps $\bar{G}_{i,j}$ isomorphically onto $G_{i-1,j}$. Note that the spaces $\{G_{1,j} \mid 1 \leqslant j \leqslant N\}$ are all in Z_1 and hence are already x-images and the spaces $\{G_{i,i} \mid 1 \leqslant i \leqslant N\}$ form the complement of $A^y \cap Z$ in Z, hence are not in K^y_{xy}, and so cannot be y-images at all. Similarly we select a space $\bar{G}'_{ij}$ for each $\{G'_{ij} \mid 1 \leqslant i < j \leqslant n\}$.

Similarly for each space $C_{s,t}$ which is part of the complement of $Z \cap Z'$ in I, we choose a space $\bar{C}_{s,t}$ so that x and y each map $\bar{C}_{s,t}$ isomorphically onto $C_{s,t}$; using the invariance of $C_{s,t}$ under R, the selection of $\bar{C}_{s,t}$ can be made so as to guarantee, as in the cases already discussed, that the diagram



commutes.

Finally, referring to Fig. 15, we note that the spaces K_{sj} corresponding to the $\bar{P}_{sj}$ in the bottom diagonal are not in K^x_{xy} and the spaces K_{sj} corresponding to the $\bar{P}_{sj}$ in the top row are not in K^y_{xy}. Thus we can complete our construction of a complementary space of $K_x + K_y$ in K_{xy} by selecting a space $\bar{K}_{ij}$ for each K_{ij} with $i > j$; as before, this can be done so that the diagram



commutes. Thus

$$\sum \oplus \{\bar{H}_{ij} \mid i \geqslant 2\} \oplus \sum \oplus \{\bar{G}_{ij} \mid i > 1\}$$
$$\oplus \sum \{\bar{G}'_{ij} \mid i < j\} \oplus \sum \oplus \{\bar{C}_{s,t}\}$$
$$\oplus \sum \{\bar{K}_{ij} \mid i > j\}$$

is the appropriate complement to $K_x + K_y$ in K_{xy}.

Finally, we must choose a relative complement to K_{xy} in A. We recall that $K_{1,0}$ is the complement of K^x_{xy} in A^x and that $K_{1,1}$ is the complement of K^y_{xy} in A^y, and that R takes $K_{1,0}$ isomorphically onto $K_{1,1}$. We therefore choose a space $\bar{K}_{1,0}$ so that x maps $\bar{K}_{1,0}$ isomorphically onto $K_{1,0}$ and y maps $\bar{K}_{1,0}$ isomorphically onto $K_{1,1}$ and the maps commute. We claim that $\bar{K}_{1,0}$ is a complement to K_{xy} in A. Indeed, let $a \in A$ and write $xa = v_1 + v_2$, where $v_1 \in K_{1,0}$ and $v_2 \in K_{xy}^x$; we obtain $xa = xa_1 + xa_2$ where $a_1 \in \bar{K}_{1,0}$ and $a_2 \in K_{xy}$. Hence $a - (a_1 + a_2) \in K_x$, so $a = a_1 + b$ where $b \in K_{xy}$; thus the claim is verified. This completes the proof of the theorem. ▮

The final step in the characterization of the $\aleph_0$-categorical groups G in our class is to show that G is $\aleph_0$-categorical iff A is an $\aleph_0$-categorical S-module. Since we prefer to discuss this in greater generality, we defer its proof to the Appendix. However, assuming its correctness, we can conclude that G is an $\aleph_0$-categorical group iff the associated linear relation $\langle A; R\rangle$ is $\aleph_0$-categorical.

This criterion is well adapted to the situation in which an Abelian by finite group G is presented explicitly in terms of a normal Abelian subgroup A of finite index in G and of bounded exponent, and generators $g_1, g_2, ..., g_k$ of G over A, assuming that for each prime p:

(a) If p divides $|G/A|$, then A has no element of order p^2, and

(b) if G/A has a noncyclic Sylow p-subgroup P, then $p = 2$ and $P = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Namely, one must choose words v, w in the $\{g_i\}$ representing generators of P (modulo A), and compute the action of v and w on A via commutation,

$$f\colon a \mapsto ava^{-1}v^{-1}, \qquad g\colon a \mapsto awa^{-1}w^{-1}$$

corresponding to the operators $1 - v$, $1 - w$ in the group ring. Finally, it must be decided, in terms of the criteria of Theorem 1, whether the structure $\langle A; R\rangle$ is $\aleph_0$-categorical, where

$$Rab \qquad \text{iff} \quad (\exists c)(f(c) = a \text{ and } g(c) = b).$$

In practice, as the examples illustrate, this tends to be a straightforward matter, even though the application of Theorem 1 cannot really be called an algorithm.

10. Concluding Remarks

The motivation for the research reported on in this article was to extend the results of [9] on $\aleph_0$-categorical Abelian by finite groups to the case where the Sylow p-subgroup of G/A was not necessarily cyclic. In the simplest case, where A has exponent 2 and $G/A \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ we have succeeded—but only because the group-theoretical and lattice-theoretical machinery is available. Thus, the lattice L of vector spaces defined in Section 6 is essentially generated by two chains and is therefore distributive, a basic fact for our analysis. The corresponding lattices in the next simplest cases, where A has exponent 2 and $G/A \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and where A has exponent 3 and $G/A \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$, do not in general have this property. Thus any further research in this direction seems to require a still more elaborate analysis than that presented here.

APPENDIX: $\aleph_0$-CATEGORICAL MODULES

Our purpose here is to justify two claims referred to in the text. Both of these claims are immediate consequences of Theorem 16, and are contained in Theorem 17.

THEOREM 16. *Suppose that G is a group with an Abelian normal subgroup A and finite factor group $F = G/A$. For each prime p, let A_p be the p-primary component of A and let F_p be a Sylow p-subgroup of F. Then the following are equivalent*:

(1) *G is $\aleph_0$-categorical.*

(2) *The $\mathbb{Z}[F]$-module A is $\aleph_0$-categorical.*

(3) *A has finite exponent and, for each prime p, the $\mathbb{Z}[F_p]$-module A_p is $\aleph_0$-categorical.*

THEOREM 17. (1) *Let G be a group with a normal Abelian subgroup A of exponent 2 such that $G/A \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and let $S = F_2[\mathbb{Z}_2 \times \mathbb{Z}_2]$. Then G is an $\aleph_0$-categorical group if and only if A is an $\aleph_0$-categorical S-module.*

(2) *Let G be a group with a normal Abelian subgroup A of square-free exponent and finite index. Assume that every Sylow p-subgroup of G/A is cyclic. Then G is $\aleph_0$-categorical.*

The proof of Theorem 16 relies on two facts concerning general $\aleph_0$-categorical modules.

THEOREM 18. *Let B be an $\aleph_0$-categorical module with submodule A. If either of the following holds then A is $\aleph_0$-categorical:*

(1) *A is a direct summand of B.*

(2) *B/A is finite.*

Theorem 18 depends heavily on the theory of $\aleph_0$-categorical modules as developed by Baur [2] and reviewed below. Before discussing this material, we will show that Theorem 16 is indeed a consequence of Theorem 18.

Proof of Theorem 16. (2) $\Rightarrow$ (1): Given that A is $\aleph_0$-categorical as a $\mathbb{Z}[F]$-

module, the same is true of a disjoint sum of a finite number of copies of A. That is, if we define a structure

$$\mathscr{P} = \langle P, A, \{f_r \mid r \in \mathbb{Z}[F]\}, A_1, A_2, \ldots, A_n, g_1, g_2, \ldots, g_n \rangle,$$

where (a) P is the disjoint union of $A, A_1, A_2, \ldots, A_n$;

(b) $\langle A, \{f_r \mid r \in \mathbb{Z}[F]\}\rangle$ is the $\mathbb{Z}[F]$ module A;

(c) each g_i is a 1–1 correspondence $A \overset{g_i}{\cong} A_i$;

then $\mathscr{P}$ is also $\aleph_0$-categorical.

Now suppose that $|G/A| = n$ and that $1, h_1, h_2, \ldots, h_n$ are a complete set of coset representatives of G/A. We can then locate G within $\mathscr{P}$ by viewing A_i as the coset Ah_i, where $g_i(a)$ is identified with ah_i. To show that G is $\aleph_0$-categorical it suffices to show that G is definable in $\mathscr{P}$, that is, that the multiplication of G is definable in $\mathscr{P}$. Now $(ah_i)(bh_j) = a(h_ibh_i^{-1})(h_ih_j)$. The map $b \to h_ibh_i^{-1}$ is represented in $\mathscr{P}$ by an element of $\mathbb{Z}[F]$, that is, $h_ibh_i^{-1} = f_{r(i)}(b)$ for all $b \in A$ for some $r(i) \in \mathbb{Z}[F]$. Also $h_ih_j = a(i,j)h_{k(i,j)}$ where $a(i,j) \in A$. It follows that the group G is definable in the structure $\langle \mathscr{P}, \{a_{ij} \mid 1 \leqslant i, j \leqslant n\}\rangle$ since the rule $x \cdot y = z$ of multiplication of G is defined by the disjunction of $(n+1)^2$ formulas of the type

$$(\exists a)(\exists b)(x = g_i(a) \wedge y = g_j(b) \wedge z = g_{k(i,j)}(a + f_{r(i)}(b) + a(i,j))).$$

Since $\langle \mathscr{P}, \{a(i,j) \mid 1 \leqslant i, j \leqslant n\}\rangle$ is $\aleph_0$-categorical, the same is true of G.

(1) $\Rightarrow$ (2): In general A will not be definable in G. If, however, we let C be the centralizer of A in G and let B be the center of C, then

$$A \subseteq B \subseteq C$$

are all normal in G and B is in fact a *definable* Abelian normal subgroup of G. To see this, it suffices to check that C is definable; but C is in fact the centralizer of a finite subset of G (since C is the centralizer of some subset of G, and $[G : C]$ is finite).

Now B may also be viewed as a $\mathbb{Z}[F]$-module, definable over G, and hence B is an $\aleph_0$-categorical $\mathbb{Z}[F]$-module. But then, by Theorem 18.2, since B/A is finite, A is an $\aleph_0$-categorical $\mathbb{Z}[F]$-module.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (2): Since A is the finite direct product of the $\mathbb{Z}[F]$-modules A_p, it suffices to show that each A_p is $\aleph_0$-categorical as a $\mathbb{Z}[F]$-module. Let A denote A_p considered as a $\mathbb{Z}[F]$-module, and let A_p denote A_p considered as a $\mathbb{Z}[F_p]$-module; A_p is $\aleph_0$-categorical by assumption.

We apply a theorem of Higman from the theory of relatively projective modules and induced representations (see Curtis and Reiner [5, Sect. 63]).

Let A_p^F be the $\mathbb{Z}[F]$-module induced from A_p. Then Higman's theorem says that since F_p is a Sylow p-subgroup of F, A is a direct summand of A_p^F. Now A_p^F is definable over A_p since it is the finite direct sum of copies of A_p with a definable F-action. Thus A_p^F inherits the $\aleph_0$-categoricity of A_p, and the same applies to A by Theorem 18.1. Thus A is an $\aleph_0$-categorical $\mathbb{Z}[F]$ module. ∎

Let us now review Baur's theory. Note the important role played by those modules satisfying the property that every finite subset is contained in a finite direct summand.

Fact 19 (*Baur* [2]). Let M be a countable module. Then the following are equivalent:

(1) M is $\aleph_0$-categorical;

(2) every finite subset of M is contained in a finite direct summand of M and M has only finitely many distinct (up to isomorphism) finite indecomposable direct summands;

(3) there are finitely many indecomposable modules M_i such that M admits a direct sum decomposition $M \simeq \oplus\sum M_{ij}$ with $M_{ij} \simeq M_i$ for $j < n_i \leqslant \omega$.

Before remarking on the proof of Fact 19, we recall that a submodule $M \subseteq N$ is *pure in* N if any system of linear equations with parameters in M which is solvable in N is already solvable in M. If M is a direct summand of N, then clearly M is pure in N. The converse is false in general; if, however, M is finite, then M pure in N implies that M is a direct summand of N. (Lemma 3 of Baur [2] strengthens this remark.) This fact is used to construct inductively, for any module satisfying (2), a decomposition satisfying (3); to continue the inductive construction, the countability of M is also necessary. To prove that (3) ⇒ (2), the same fact is used, together with the Krull–Schmidt Theorem on uniqueness of decompositions (see Curtis and Reiner [5]). That (3) ⇒ (1) is clear. The proof that (1) ⇒ (2) is a central argument in Baur [2] and our proof of Theorem 18.1 is patterned on and requires familiarity with Baur's argument. We prove Theorem 18.1 in the following stronger form.

THEOREM 20. *Let N be an $\aleph_0$-categorical module and let $M \subseteq N$ be a pure submodule. Then M is $\aleph_0$-categorical.*

Proof. By the Lowenheim–Skolem Theorem it suffices to prove this for the case where N is countable. We claim that M then satisfies condition (2) of Fact 19 and hence is $\aleph_0$-categorical.

Let I be a finite indecomposable direct summand of M. Baur showed that if N is $\aleph_0$-categorical, then any finite subset of N is contained in a finite direct summand of N; in particular, $I \subseteq J$ where J is a finite direct summand of N. Since I is pure in M and M is pure in N, it follows that I is pure in N, and hence is pure in J. By the fact mentioned earlier, I is a direct summand of J and hence of N. Thus every finite indecomposable direct summand of M is also a direct summand of N; since N is $\aleph_0$-categorical, there can be only finitely many such I, up to isomorphism.

Thus it suffices to show that every finite subset $M_0 \subseteq M$ is contained in a finite direct summand of M. As in Baur's argument, it suffices to find a number k such that any system of linear equations with parameters in M_0 which is solvable in M has a solution $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ containing at most k distinct elements of M. This property is, however, inherited from N, where it follows from $\aleph_0$-categoricity, as shown in Baur [2]. Indeed, if Γ is a linear system defined over M_0, as above, and solvable in M, then it is also solvable in N, and so has a solution in N with at most k different elements. If we specialize Γ to a system Γ' involving altogether only k distinct variables, then the solvability of Γ' in N entails its solvability in M (by purity); this gives a solution to Γ in M having only k distinct elements of M. Thus the bound k for M may be taken to be equal to the given bound for N.

Hence M satisfies condition (2) of Fact 19 and hence is $\aleph_0$-categorical. ▌

Examples of non-$\aleph_0$-categorical submodules of $\aleph_0$-categorical modules abound. For example, any $\mathbb{Z}_n[1]$-module is $\aleph_0$-categorical (since it is essentially an Abelian-by-cyclic group of the type considered in [9]) as is any module induced from a $\mathbb{Z}_n[1]$-module. But every module M over a group algebra $\mathbb{Z}_n[H]$ of a finite group H is a submodule of a module induced from a $\mathbb{Z}_n[1]$-module. (This observation arises in the context of Higman's Theorem cited above.)

We will now prove a result slightly stronger than Theorem 18.2.

THEOREM 21. *Let M be an $\aleph_0$-categorical module, A a finite Abelian group, and $h: M \to A$ a group homomorphism. Then the structure $\langle M, A, h\rangle$ (where, of course, M is equipped with module operations) is $\aleph_0$-categorical.*

Proof. Note in particular that given a short exact sequence $0 \to M' \to M \xrightarrow{\pi} M'' \to 0$ with M' $\aleph_0$-categorical and M'' finite, then $M'' = \ker \pi$ is definable and hence $\aleph_0$-categorical; this is Theorem 18.2.

We may assume, without loss of generality, that M is countable. Then, by Baur's Theorem, there are finitely many finite modules M_i and a decomposition $M = \oplus \sum M_{ij}$ with $M_{ij} \simeq M_i$ for each j, $1 \leqslant j < n_i \leqslant \omega$. Define $\mathfrak{A}_{ij} = \langle M_{ij}, A, h_{ij}\rangle$ where h_{ij} is h restricted to M_{ij}. There are only a finite number of isomorphism types among the structures $\mathfrak{A}_{ij}$, and, after a change of notation, we may assume that for each fixed i the structures $\mathfrak{A}_{ij}$ are all isomorphic to a structure $\mathfrak{A}_i = \langle M_i, A, h_i\rangle$. For each i let $\mathfrak{B}_i = \langle M^*_i, A, h^*_i\rangle$ where $M^*_i = M_i^{(n_i)}$ and $h^*_i(m^*) = \sum_j h_i(m^*_j)$ for $m^* \in M^*_i$. Since $\langle M, A, h\rangle \simeq \langle \oplus\sum M^*_i, A, \oplus\sum h^*_i\rangle$, it suffices to show that each $\mathfrak{B}_i$ is $\aleph_0$-categorical.

As a preliminary step we construct a number of automorphisms of $\mathfrak{B}_i$ which will be used to obtain a bound on the number of n-types of $\mathscr{B}_i$. Let r be the characteristic of the base ring R (which we assume to be finite), let $s \equiv 1 \pmod{r}$, let $j_0 < n_i$, and let J be a set of $s - 1$ distinct integers each less than n_i but not including j_0. Given s, j_0, and J, we define an automorphism θ of M^*_i by

$$[\theta(m^*)]_j = m_j^* \qquad \text{if} \quad j \notin J,$$

$$= m_j^* - m^*_{j_0} \qquad \text{if} \quad j \in J.$$

Evidently θ is a module isomorphism and $h^*_i(\theta(m^*)) = h^*_i(m^*) - (s-1)\, h_i(m^*_{j_0}) = h^*_i(m^*)$ so that θ is an automorphism of $\mathscr{B}$ which fixes A.

Using these automorphisms it is easy to show, by induction on n, that there is a number $f(n)$ such that any n-tuple t of elements of M^*_i can be mapped, by an automorphism of $\mathscr{B}_i$, to an n-tuple t' of elements of M^*_i, each of which has all of its nonzero coordinates in the first $f(n)$ places. Thus the number of n-types of elements of M^*_i is bounded for each i and hence $\mathscr{B}_i$ is $\aleph_0$-categorical, as was to be proved. ∎

We have now completed the development of the relevant supplementary module-theoretic information bearing on $\aleph_0$-categorical Abelian by finite groups. We conclude with an additional result in the same vein, which we present for its own intrinsic interest.

THEOREM 22. *If* $0 \to M' \to M \xrightarrow{\pi} M'' \to 0$ *is a short exact sequence of* R*-modules with* M' *finite and* M'' $\aleph_0$*-categorical, then* M *is* $\aleph_0$*-categorical.*

Proof. Take M countable and R finite. If we view M', M, and M'' as Abelian groups, then, by the $\aleph_0$-categoricity of M'', M'' and hence also M are of bounded order; hence, by the structure theory of Abelian groups of bounded order, M' can be extended to a finite direct summand M_1 of M. Thus, as an Abelian group, we can write $M = M_1 \oplus M_2$, where $M_1 \supseteq M'$ and is finite. Then if we define $M''_1 = \pi(M_1)$ and $M''_2 = \pi(M_2)$, we see that $M''_1 \simeq M_1/M'$ and $M''_2 \simeq M_2$ and $M'' = M''_1 \oplus M''_2$.

Since M'' is an $\aleph_0$-categorical module, the finite subgroup M''_1 can be extended to a finite *module* direct summand M''_3 of M''. Thus we can write $M'' = M''_3 \oplus M''_4$, where M''_3 and M''_4 are submodules of M'', and by Theorem 18, M''_4 is an $\aleph_0$-categorical module. Note also that if we define $M_3 = \pi^{-1}[M''_3]$, then $M_3 \supseteq M_1$. Furthermore, the composite homomorphism

$$\chi\colon M''_4 \hookrightarrow M'' \xrightarrow{\pi_2} M''_2 \overset{\pi^{-1}}{\simeq} M_2$$

is a monomorphism of Abelian groups, so that $M_4 = \chi(M''_4)$ is a subgroup of M_2. Actually, since M''_4 is a direct summand of M'', M''_4 is pure in M'' and hence is pure in M''_2. It follows that M_4 is a pure subgroup of M_2. Since M_4 is a pure subgroup of M_2 and M_4 is a group of bounded order, it follows that M_4 is a direct summand of M_2 (see Kaplansky [8]) so that we can write $M_2 = M_4 \oplus M_5$ as groups. Thus we have $M = M_6 \oplus M_4$, where $M_6 = M_1 \oplus M_5$.

Summarizing, we have shown that M has a subgroup M_4 which is a direct summand of M of finite index; furthermore the submodule M''_4 of M'' of finite index which is $\aleph_0$-categorical is, as a group, isomorphic to M_4. Although M_4 is not necessarily a submodule of M, for each $\chi(m) \in M_4$, $r\chi(m)$ is defined and $\pi(r\chi(m)) = rm$. On the other hand, for $m \in M''_4$, $rm \in M''_4$ so that $\chi(rm) \in M_4$ and $\pi(\chi(rm)) = rm$. Hence $r\chi(m) - \chi(rm) \in \ker(\pi)$. Thus for each $r \in R$ we can define $f_r \colon M''_4 \to M_6$ by $f_r(m) = r\chi(m) - \chi(rm)$. This gives $r\chi(m) = f_r(m) + \chi(rm)$, so that scalar multiplication on M_4 can be recovered from $\{f_r \mid r \in R\}$. These maps f_r are additive, and may be combined into a single map

$$(\times f_r) \circ \nabla \colon M''_4 \to M_6{}^N,$$

where $N = \operatorname{card}(R)$ and $\nabla(m) = (m, m, \ldots, m)$ is the diagonal map. By the previous theorem, the structure

$$\mathfrak{A}_1 = \langle M''_4, M_6{}^N, \times f_r \circ \nabla\rangle$$

is $\aleph_0$-categorical, from which it follows that

$$\mathfrak{A} = \langle M''_4, M_6; \{f_r \mid r \in R\}\rangle$$

is $\aleph_0$-categorical. From $\mathfrak{A}$ we can recover the module structure of M by taking

(1) $M = M_6 \oplus M''_4$ as Abelian groups;

(2) $r(m_6 + m''_4) = rm_6 + (f_r(m''_4) + rm''_4)$.

Here, since M_6 is finite and R is finite, the products rm_6 are given explicitly, $f_r(m''_4)$ is computed in $\mathfrak{A}$, and rm''_4 is computed in the module M''_4 in $\mathfrak{A}$.

Thus the module M is definable over $\mathfrak{A}$ and is therefore $\aleph_0$-categorical. ∎

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