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# $\aleph_0$-Categoricity for Rings without Nilpotent Elements and for Boolean Structures*

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## 0. INTRODUCTION

In this paper we give a complete classification of the complete $\aleph_0$-categorical theories of rings with 1 and without nonzero nilpotent elements.

It is not feasible to state the main result at the outset, so we simply give a sketch of our analysis.

First we show that if $R$ is any ring with Th($R$) $\aleph_0$-categorical then the additive group of $R$ has bounded order, and there exists a monic $f$ in $Z[x]$ which vanishes identically on $R$. Both results are consequences of Ryll-Nardzewski's theorem [19].

Next, we reduce the general case to the case of $R$ whose additive group is a $p$-group for some prime $p$. The main tool here is Grzegorczyk's theorem [7, 22] on products of $\aleph_0$-categorical theories.

Next, we suppose $R$ has no nonzero nilpotent elements. Then $R$ has characteristic $p$, so is an algebra over the field $F_p$ of $p$ elements. But $R$

* The main theorems of this paper were proved independently by the two authors. This article is based on Macintyre's 1972–1973 draft. Rosenstein's 1971 draft also contained a proof that the categories of countable subrings of $\prod_{i \in I} F_i(p^n)$ and $\prod_{i \in J} F_i(q^m)$ are equivalent whenever the lattices of divisors of $m$ and $n$ are isomorphic (where $F_k(t)$ is the field with $t$ elements); this generalizes a theorem of R. W. Stringall (The categories of $p$-rings are equivalent, *Proc. Amer. Math. Soc.* **29** (1971), pp. 229–235).

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satisfies the polynomial identity $f = 0$ described above, so by a theorem of McCoy [14] $R$ is commutative.

So by now we are considering commutative algebras $R$ over $F_p$, satisfying a polynomial identity $f = 0$, with a unit and without nonzero nilpotent elements. Arens and Kaplansky [1] gave an important structure theorem for *countable* $R$ satisfying these conditions. Their theorem is a generalization of Stone's theorem [8]. To $R$ they associate a Boolean space $X$, a finite sequence $X_i$, $i < n$, of closed subspaces of $X$, a finite field $F$, and a sequence $F_i$, $i < n$, of subfields of $F$. They give $F$ the discrete topology, and consider $C(X, F; X_i, i < n; F_i, i < n)$, the ring of continuous functions $g: X \to F$ such that $g(X_i) \subseteq F_i$ for $i < n$. Then

$$R \cong C(X, F; X_i, i < n; F_i, i < n). \tag{*}$$

The problem is to find out what the $\aleph_0$-categoricity of Th($R$) tells us about $X$ and $X_i$, $i < n$. The key idea is to use Stone duality. Let $B$ be the dual algebra of $X$. Then $X_i$, $i < n$, correspond to ideals $I_i$, $i < n$, of $B$. So to $R$ we have associated a relational system $\mathscr{A}(R)$ consisting of the Boolean algebra $B$ with distinguished ideals $I_i$, $i < n$. We show how to to interpret $\mathscr{A}(R)$ in $R$, and conclude that if Th($R$) is $\aleph_0$-categorical then Th($\mathscr{A}(R)$) is $\aleph_0$-categorical.

Then we prove a general theorem, which can be construed as a relative of a theorem of Waskiewicz and Weglorz [22], which enables us to conclude that, for $R$ satisfying (*), if Th($\mathscr{A}(R)$) is $\aleph_0$-categorical then Th($R$) is $\aleph_0$-categorical. So we have reduced our problem to one about $\aleph_0$-categoricity of systems consisting of a Boolean algebra and a finite sequence of distinguished ideals. Finally we solve this problem about Boolean algebras.

Other articles in the literature dealing with $\aleph_0$-categoricity are [2, 6, 12, 17, 18, 22].

## 1. Model–Theoretic Preliminaries

1.1. We assume familiarity with the basic material of model theory up to the Ryll-Nardzewski theorem [19]. A good reference is [20].

We will begin by listing some known general results about $\aleph_0$-categoricity, and then add one new result to the list.

Throughout, $L$ will be a countable first-order language and $T$ will be an $L$-theory. Usually $T$ will be complete, and then $S_n(T)$ will be the space of complete $n$-types over $T$.

It is convenient to make the definition that an $L$-structure $\mathscr{M}$ is $\aleph_0$-categorical if Th($\mathscr{M}$) is $\aleph_0$-categorical. Note that any finite $\mathscr{M}$ is $\aleph_0$-categorical.

1.2. The fundamental theorem is

THEOREM 1 (Ryll-Nardzewski [19]). *Suppose $T$ is complete. Then $T$ is $\aleph_0$-categorical if and only if $S_n(T)$ is finite for each $n$.*

In algebraic investigations the following is very useful.

THEOREM 2 (Grzegorcyck [7, 22]). *Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are $\aleph_0$-categorical. Then $\mathcal{M}_1 \times \mathcal{M}_2$ is $\aleph_0$-categorical.*

1.3. Grzegorcyzk's theorem should be thought of as a preservation theorem. Another important preservation theorem in this direction was found by Waskiewicz and Weglorz [22].

Suppose $X$ is a topological space, and $\mathcal{M}$ an $L$-structure. Give $\mathcal{M}$ the discrete topology, and consider the set of continuous functions $f: X \to \mathcal{M}$. It is easily seen that this set forms a substructure of the $L$-structure $\mathcal{M}^X$. In this way we get an $L$-structure $C(X, \mathcal{M})$.

For general $X$, little is known about the model theory of $C(X, \mathcal{M})$. When $X$ is Boolean, however, the situation is much better understood [5]. When $\mathcal{M}$ is finite, and $X$ Boolean, the structures $C(X, \mathcal{M})$ are just a dual version of the Boolean extensions [2].

In order to make natural the theorem of Waskiewicz and Weglorz, we have to look at $\aleph_0$-categorical theories of Boolean algebras.

THEOREM 3 (Folklore). *A Boolean algebra $B$ is $\aleph_0$-categorical if and only if $B$ has only finitely many atoms.*

*Proof.* Sufficiency is well known.

For necessity, we apply Theorem 1. For each $n < w$, consider the formula $\Phi_n(v_0)$ which expresses that $v_0$ is a join of $n$ atoms. $\Phi_n$ and $\Phi_m$ are incompatible relative to the theory of Boolean algebras, if $n \neq m$. So if $B$ has infinitely many atoms, $S_1(\mathrm{Th}(B))$ is infinite, and so, by Theorem 1, $B$ is not $\aleph_0$-categorical. ∎

Now we look back at $C(X, \mathcal{M})$, where $X$ is Boolean. Suppose $\mathcal{M}$ is $\aleph_0$-categorical. When is $C(X, \mathcal{M})$ $\aleph_0$-categorical? Now, with $X$ we have associated $B(X)$, its dual algebra of clopen sets. Waskiewicz and Weglorz proved

THEOREM 4 (Waskiewicz and Weglorz [22]). *Suppose $X$ is Boolean, and $B(X)$ is $\aleph_0$-categorical. Suppose $\mathcal{M}$ is $\aleph_0$-categorical. Then $C(X, \mathcal{M})$ is $\aleph_0$-categorical.*

COROLLARY. *Suppose $X$ is Boolean, with only finitely many isolated points. Suppose $\mathcal{M}$ is $\aleph_0$-categorical. Then $C(X, \mathcal{M})$ is $\aleph_0$-categorical.*

*Proof.* By Stone duality, $B(X)$ has $n$ atoms if and only if $X$ has $n$ isolated points. Now apply Theorem 3. ∎

1.4. To analyze the $\aleph_0$-categorical theories of rings, we need a generalization of the construction used in 1.3.

*Let $X$ be Boolean, and $\mathcal{M}$ an L-structure. Let $X_i$, $i < n$, be closed subsets of $X$, and $\mathcal{M}_i$, $i < n$, substructures of $\mathcal{M}$. Suppose that the map $X_i \mapsto \mathcal{M}_i$ is order-preserving. Then it is clear that we get a substructure of $C(X, \mathcal{M})$ if we restrict to those $f$ in $C(X, \mathcal{M})$ such that $f(X_i) \subseteq \mathcal{M}_i$ for $i < n$. In this way we get a structure $C(X, \mathcal{M}; X_i, i < n; \mathcal{M}_i, i < n)$.*

For this type of structure we will produce an analog of Theorem 4. In order to formulate it, we need to use Stone duality.

For each closed $X_i$ above, define

$$\hat{X}_i = \{a \in B(X) \colon a \cap X_i = 0\}.$$

Then $\hat{X}_i$ is an ideal in $B(X)$.

In this way we get a system $\mathcal{A}(X; X_i, i < n)$ consisting of the Boolean algebra $B(X)$ and the distinguished ideals $\hat{X}_i$, $i < n$.

THEOREM 5. *Suppose $X$ is Boolean, and $X_i$, $i < n$, are closed subsets of $X$. Suppose $\mathcal{M}$ is a finite L-structure, and $\mathcal{M}_i$, $i < n$, are substructures of $\mathcal{M}$. Suppose $\mathcal{A}(X; X_i, i < n)$ is $\aleph_0$-categorical. Then $C(X, \mathcal{M}; X_i, i < n; \mathcal{M}_i, i < n)$ is $\aleph_0$-categorical.*

*Proof.* We shall just outline the proof. It is sufficient to show that $C(X, \mathcal{M}; X_i, i < n; \mathcal{M}_i, i < n)$ is interpretable in $\mathcal{A}(X; X_i, i < n)$, for then the result is immediate from Theorem 1.

The key point is the finiteness of $\mathcal{M}$. Let $k$ be the cardinality of $\mathcal{M}$. Let $m_0, \ldots, m_{k-1}$ be an enumeration of $\mathcal{M}$. Suppose $f \in C(X, \mathcal{M})$. Then to $f$ we associate the $k$-tuple $\langle f^{-1}(m_0), \ldots, f^{-1}(m_{k-1})\rangle$ of clopen subsets of $X$. These clopen sets form a partition of $X$.

Conversely, let $b_0, \ldots, b_{k-1}$ be elements of $B(X)$ such that $b_0 \cup \cdots \cup b_{k-1} = 1$, and $b_i \cap b_j = 0$ if $i \neq j$. Then there is a unique element $f$ of $C(X, \mathcal{M})$ such that $f^{-1}(m_i) = b_i$ for $i < k$.

Thus we can interpret the elements of $C(X, \mathcal{M})$ as $k$-tuples $\langle b_0, \ldots, b_{k-1}\rangle$ of elements of $B(X)$ satisfying the conditions of the previous paragraph.

Because of the finiteness of $\mathcal{M}$, it is clear that the relational and operational structure of $C(X, \mathcal{M})$ is interpretable in $B(X)$.

Finally, $\langle b_0, \ldots, b_{k-1}\rangle$ corresponds to an element of $C(X, \mathcal{M}; X_i, i < n; \mathcal{M}_i, i < n)$ if and only if the following condition is satisfied. If $m_i \notin \mathcal{M}_j$ then $b_i \in \hat{X}_j$.

It follows that $C(X, \mathscr{M}; X_i, i < n; \mathscr{M}_i, i < n)$ is interpretable in $\mathscr{A}(X; X_i, i < n)$, and the theorem is proved. ∎

*Notes.* (1) This proof is based on a remark of Dana Scott. An earlier proof used ideas related to Comer's [5].

(2) We do not know what the possibilities are if $\mathscr{M}$ is infinite.

(3) Using Stone duality, it is clear that there is a natural correspondence between systems consisting of a Boolean algebra $B$ with distinguished ideals $I_i$, $i < n$, and Boolean spaces $X$ with closed subsets $X_i$, $i < n$.

(4) Theorem 4 raises the problem: Classify the systems $(B; I_i, i < n)$ which are $\aleph_0$-categorical.

This problem is discussed in Section 4. The answer is a generalization of Theorem 2. Systems $(B; I_1, ..., I_n)$ have been treated by Rabin in [16], but we do not see how to get information on $\aleph_0$-categoricity by this approach.

## 2. Ring Theory

2.1. We formalize ring theory in the usual language $L$ with $+$, $\cdot$, 0. We shall be trying to classify rings $R$ which are $\aleph_0$-categorical (relative to $L$).

We shall make a cumulative series of assumptions about $R$. In a given subsection 2.–, the lemmas will be proved relative to all assumptions made previously.

2.2. *Assumption*: $R$ is $\aleph_0$-categorical.

Lemma 1. *There exists an integer $n > 0$ such that for all $x$ in $R$, $nx = 0$.*

*Proof.* Consider the formulas $\Phi_m(v_0, v_1)$:

$$v_0 = \underbrace{v_1 + v_1 + \cdots}_{m \text{ times}}$$

By Theorem 1, there exists $N$ such that

$$R \models (\forall v_0, v_1)\Big[\Phi_{N+1}(v_0, v_1) \leftrightarrow \bigvee_{j \leqslant N} \Phi_j(v_0, v_1)\Big].$$

Thus,

$$R \models (\forall v_1)[nv_1 = 0], \qquad \text{where} \quad n = N! \ ∎$$

LEMMA 2. *There exists an integer $N > 0$ such that for each $x$ in $R$ either*

$$x^{N+1} = x^N,$$

*or*

$$x^{N+1} = x^{N-1},$$

*or*

$$\ldots,$$

*or*

$$x^{N+1} = x.$$

*Proof.* This is essentially the same idea as in Lemma 1, and we omit the details. ∎

COROLLARY. *There is a monic $f \in Z[x]$ such that $f(r) = 0$ for all $r$ in $R$.*

*Proof.* Let $f(x) = \prod_{1 \leqslant j \leqslant N} (x^{N+1} - x^j)$. ∎

2.3. Now we decompose $R$ as a product of ideals. Select $n > 0$ as in Lemma 1, and let $n = \prod_i p_i^{\alpha_i}$ be the prime factorization of $n$. Let $R_i = \{x \in R: p_i^{\alpha_i} x = 0\}$. Then $R_i$ is an ideal in $R$, and by familiar arguments $R \cong \prod_i R_i$.

LEMMA 3. *Each $R_i$ is $\aleph_0$-categorical.*

*Proof.* Each $R_i$ is definable in $R$. Now apply Theorem 1. ∎

*Reduction of the Problem.* By Theorem 2, $R$ will be $\aleph_0$-categorical provided each $R_i$ is $\aleph_0$-categorical. So we may now confine ourselves without loss of generality to rings $R$ for which there is a prime power $p^k$ such that $p^k x = 0$ for all $x$ in $R$.

*Note.* $R$ has a 1 if and only if each $R_i$ has a 1.

2.4. *Assumption*: $R$ has no nonzero nilpotent elements.

LEMMA 4. (i) *$px = 0$ for all $x$ in $R$.* (ii) *For some $j$, $x^{p^j} = x$ for all $x$ in $R$.* (iii) *$R$ is commutative.*

*Proof.* (i) $p^2 y = 0 \Rightarrow (py)^2 = 0 \Rightarrow py = 0$ since there are no nonzero idempotents.

(ii) Fix $x$ in $R$, and consider the subring $S$ generated by $x$. $S$ is commutative and without nilpotent elements, so semisimple. By Lemma 2, and (i), $S$ is finite, so artinian. So $S$ is a finite product of finite fields, so for some $j$, $x^{p^j} = x$.

(iii) This follows from (ii) by [9, p. 73]. ∎

2.5. Now we can apply a theorem of Arens and Kaplansky [1] and get to the heart of the matter. In [1, Theorem 8.1, Corollary], they proved:

Suppose $\mathscr{A}$ is a ring of characteristic $p$ in which every element satisfies $a^{p^n} = a$, and in which every ideal is countably generated. There there exists a locally compact zero-dimensional space $X$, with a closed subset $X_k$ for each divisor $k$ of $n$, such that $\mathscr{A}$ is isomorphic to the ring of all continuous functions from $X$ to $F_{p^n}$, vanishing outside a compact set, and on $X_k$ taking values in $F_{p^k}$.

Suppose $\mathscr{A}$ has 1. Then by a routine argument $X$ must be compact and so Boolean.

2.6. *Assumptions*: $R$ has 1. $R$ is countable.

LEMMA 5. *$R$ is of the form $C(X, F;\ X_i, i < n;\ F_i, i < n)$, where $X$ is Boolean, $F$ is a finite field, the $F_i$ are subfields of $F$, and each $X_i$ is closed.*

*Proof.* By Arens and Kaplansky [1]. ∎

2.7. Now suppose $R$ is of the form given in Lemma 6. For $i < n$, define

$$Y_i = \{x \in X: f(x) \in F_i,\ \forall f \in R\}.$$

Since $F$ is finite, each $Y_i$ is closed. Clearly $X_i \subseteq Y_i$. It is readily seen that that $R = C(X, F;\ Y_i, i < n;\ F_i, i < n)$.

So we can now assume without loss of generality that

$$R = C(X, F;\ X_i, i < n;\ F_i, i < n),$$

where $X_i = \{x \in X: f(x) \in F_i,\ \forall f \in R\}$.

Define $\mathscr{A}(R)$ to be the system consisting of the Boolean algebra $B(X)$ with the distinguished ideals $\hat{X}_i$, $i < n$.

LEMMA 6. *$\mathscr{A}(R)$ is interpretable in $R$.*

*Proof.* Let $E(R)$ be the Boolean algebra of idempotents of $R$ [9]. $E(R)$ is isomorphic to $B(X)$ via the correspondence $\phi$ $e \mapsto \{x \in X: e(x) = 1\}$.

What does it mean for $e$ to be in $\phi^{-1}(\hat{X}_j)$?

Suppose $b \in \hat{X}_j$, and $b \neq 0$. For each point $t$ in $b$ there exists $f_t$ in $R$ such that $f_t(t) \notin F_j$. On some clopen $b_t \subseteq b$, with $t \in b_t$, $f_t$ takes only the value $f_t(t)$. By compactness, $b$ is covered by finitely many of the $b_t$.

Let $N$ be the cardinality of $F$. There are at most $N$ possible values for $f_t(t)$. Let $T$ be a finite subset of $b$ such that $b = \bigcup_{t\in T} b_t$. Let $\alpha \in F$. Let $T_\alpha = \{t \in T: f_t(t) = \alpha\}$. Let $\beta_\alpha = \bigcup_{t\in T_\alpha} b_t$. Then it is a routine argument (cf. [1]) to show that there exists $g_\alpha$ in $R$ such that $g_\alpha$ is identically zero outside $\beta_\alpha$ and $g_\alpha$ takes the value $\alpha$ identically on $\beta_\alpha$.

We have $b = \bigcup_{\alpha \in F} \beta_\alpha$ and with each $\beta_\alpha$ we have associated $g_\alpha$. Note that if $\beta_\alpha \neq 0$ then $\alpha \notin F_j$. Let $q$ be the cardinality of $F_j$. Then $\alpha \in F_j$ if and only if $\alpha^q = \alpha$.

Suppose $\beta_\alpha \neq 0$. Then $\alpha \notin F_j$, so $\alpha^q \neq \alpha$. Define $h_\alpha$ on $X$ by

$$h_\alpha(x) = (\alpha^q - \alpha)^{-1} \quad \text{if} \quad x \in \beta_\alpha ;$$
$$h_\alpha(x) = 0 \quad \text{if} \quad x \in \beta_\alpha' .$$

Then $h_\alpha \in R$. Clearly $h_\alpha(g_\alpha{}^q - g_\alpha)$ is the characteristic function $\chi_\alpha$ of $\beta_\alpha$.

Suppose $\phi(e) = b$. Then $e = \bigcup_\alpha \phi^{-1}(\beta_\alpha) = \bigcup_\alpha \chi_\alpha$. We have $g_\alpha \chi_\alpha = g_\alpha$, and $h_\alpha(g_\alpha{}^q - g_\alpha) = \chi_\alpha$.

Thus we have shown:

> If $e \in \phi^{-1}(\hat{X}_j)$ then $e$ is of the form $\bigcup_{k<N} e_k$, where each $e_k$ is an idempotent, and there exist $u_k$, $v_k$ in $R$ such that $v_k(u_k{}^q - u_k) = e_k$. (#)

Conversely suppose $e \in E(R)$ and there exist $e_k$ $(k < N)$ in $E(R)$, and $u_k$, $v_k$ in $R$, so that (#) is satisfied. Suppose $t \in \phi(e)$ and $t \in X_j$. Then for some $k$, $t \in \phi(e_k)$. $e_k$ is of course the characteristic function of $\phi(e_k)$. Since $t \in X_j$, $u_k(t) \in F_j$. Thus $u_k{}^q(t) = u_k(t)$. Since $v_k(u_k{}^q - u_k) = e_k$, $e_k(t) = 0$, so $t \in \phi(e_k)$. This is a contradiction. Thus $\phi(e) \cap X_j = 0$, so $e \in \phi^{-1}(\hat{X}_j)$.

We have shown that $\phi^{-1}(\hat{X}_j)$ is first-order definable in $R$. The lemma is proved. ∎

2.8. A direct consequence of Lemma 6 is

LEMMA 7. *$\mathscr{A}(R)$ is $\aleph_0$-categorical.*

*Proof.* Theorem 1 and Lemma 6. ∎

But conversely we have

LEMMA 8. *Suppose $R$ is of the form $C(X, F; X_i, i < n; F_i, i < n)$, where $X$ is Boolean, $F$ is a finite field, and the $F_i$ are subfields of $F$. Let $\mathscr{A}(R)$ be the system consisting of $B(X)$ with the distinguished ideals $X_i$, $i < n$. Suppose $\mathscr{A}(R)$ is $\aleph_0$-categorical. Then $R$ is $\aleph_0$-categorical.*

*Proof.* Immediate from Theorem 5. ∎

2.9. We have now solved our main problem.

THEOREM 6. *Suppose $R$ is a countable ring with 1 and no nonzero nilpotent elements. Then, $R$ is $\aleph_0$-categorical if and only if $R$ is of the form $R_1 x, \ldots, x R_n$, where each $R_j$ is of the form $C(X_j, F_j; X_{ji}, i < n_j; F_{ji}, i < n_j)$, where each $X_j$ is Boolean, each $F_j$ is a finite field, and each $F_{ji}$ is a subfield of $F_j$,* and *each system $(B(X_j), \hat{X}_{ji}, i < n_j)$ is $\aleph_0$-categorical.*

*Proof.* Necessity. 2.3, Lemma 5, Lemma 7.

Sufficiency. Lemma 8, Theorem 2. ▌

*Notes.* (1) We can classify the systems $(B, I_j, j < n)$ which are $\aleph_0$-categorical (see Sect. 3).

(2) At present without Section 4, we can classify the systems $(B, I_j, j < n)$ when $n = 0$. This is Theorem 3. This yields examples of $\aleph_0$-categorical theories of rings, e.g., $\mathrm{Th}(C(X, F))$, where $X$ has only finitely many isolated points, and $F$ is a finite field.

(3) By Löwenheim-Skolem, there is no loss of generality, in classifying $\aleph_0$-categorical $\mathrm{Th}(R)$, in assuming that $R$ is countable.

## 3. $\aleph_0$-CATEGORICITY FOR BOOLEAN ALGEBRAS WITH DISTINGUISHED IDEALS

3.1. We are dealing with systems

$$\mathcal{M} = \langle M, \cap, \cup, ', 0, 1, J_i, i < n \rangle,$$

where $\langle M, \cap, \cup, ', 0, 1 \rangle$ is a Boolean algebra $\mathcal{M}_0$, and the $J_i$ are ideals in $\mathcal{M}_0$. Such a system will be called an augmented Boolean algebra of rank $n$.

We will make fundamental use of Stone's work [8]. To begin with, the equivalence of Boolean rings and Boolean algebras will be needed. Thus any Boolean algebra can be canonically converted to a Boolean ring, and conversely [8]. The main point for us is the functoriality of the construction, and the fact that the notions of homomorphism and ideal are independent of whether we are in the algebra or ring setup. Note, too, that a Boolean ring is naturally an algebra over $F_2$. Towards the end of this section we shall use the topological duality.

Our problem is: Classify all augmented Boolean algebras $\mathcal{M}$ such that $\mathcal{M}$ is $\aleph_0$-categorical.

### 3.2. *Limitations on $\mathcal{M}$.*

In this subsection, $\mathcal{M}$ is an augmented Boolean algebra of rank $n$, as above, and $\mathcal{M}_0$ is its underlying Boolean algebra. We assume $\mathcal{M}$ is $\aleph_0$-categorical.

LEMMA 9. *$\mathcal{M}_0$ has only finitely many atoms.*

*Proof.* See proof of Theorem 3. ∎

LEMMA 10. *For each $i$, $\mathcal{M}_0/J_i$ has only finitely many atoms.*

*Proof.* $\mathcal{M}_0/J_i$ is interpretable in $\mathcal{M}_0$. Now apply Theorem 3. ∎

These necessary conditions on $\mathcal{M}$ will not be sufficient for $\aleph_0$-categoricity. We now find a final necessary condition, which will also give us a sufficient condition for $\mathcal{M}$ to be $\aleph_0$-categorical.

Recall that a *Heyting algebra* (or Brouwer lattice) is a structure $\langle H, \wedge, \vee, \rightarrow\rangle$ such that $\langle H, \wedge, \vee\rangle$ is a lattice. $a \wedge (a \rightarrow b) \leqslant b$, and $a \rightarrow b = \bigvee \{x: x \wedge a \leqslant b\}$, for all $a, b$ in $H$ (see [3]).

The basic point for our purposes is that the set of ideals of a Boolean algebra forms a Heyting algebra. Thus, suppose $I_1$ and $I_2$ are ideals in the Boolean algebra $B$. We define

$$I_1 \wedge I_2 = I_1 \cap I_2 ;$$
$$I_1 \vee I_2 = I_1 + I_2 ;$$
$$I_1 \rightarrow I_2 = \{x \in B: xI_1 \subseteq I_2\}.$$

It can easily be verified that with these operations the set of ideals of $B$ forms a Heyting algebra. Moreover, the Heyting algebra has a 0 and 1, namely, the ideals 0 and $B$, respectively.

Now let $H(\mathcal{M})$ be the Heyting algebra of ideals of $\mathcal{M}_0$, and let $H_0(\mathcal{M})$ be the subalgebra of $H(\mathcal{M})$ generated by 0, 1, $J_i$, $i < n$.

LEMMA 11. *$H_0(\mathcal{M})$ is finite, and for each $J$ in $H_0(\mathcal{M})$, $\mathcal{M}_0/J$ has only finitely many atoms.*

*Proof.* Each member of $H_0(\mathcal{M})$ is first-order definable in $\mathcal{M}$. So if $H_0(\mathcal{M})$ were infinite, Th($\mathcal{M}$) would have infinitely many 1-types, contradicting Theorem 1. If $J \in H_0(\mathcal{M})$, $\mathcal{M}_0/J$ is interpretable in $\mathcal{M}$, so apply Theorem 3. ∎

We can now state the main result.

THEOREM 7. *Th($\mathcal{M}$) is $\aleph_0$-categorical if and only if $H_0(\mathcal{M})$ is finite and for each $J$ in $H_0(\mathcal{M})$, $\mathcal{M}_0/J$ has only finitely many atoms.*

We have proved necessity. Sufficiency will take some time.

### 3.3. *Extension of Isomorphisms*

We now have to prove some isomorphism theorems for augmented Boolean algebras. As always with $\aleph_0$-categoricity, the problem is: Given a monomorphism $f: \mathcal{M}_1 \to \mathcal{M}_2$, where $\mathcal{M}_1$ is finite, and given a finite extension $\mathcal{M}_1'$ of $\mathcal{M}_1$, extend $f$ to a monomorphism $g: \mathcal{M}_1' \to \mathcal{M}_2$.

We have to find necessary and sufficient conditions for the existence of $g$.

#### 3.3.1. *The basic lemmas.*

LEMMA 12. *Let $B$ be a finite Boolean algebra, and $B_1$ a Boolean algebra which is a proper extension of $B$. Then some atom of $B$ is not an atom of $B_1$.*

*Proof.* Let $a_1, \ldots, a_n$ be the atoms of $B$. Then $1 = a_1 \cup \cdots \cup a_n$. Let $x \in B_1$. Then $x = x \cap 1 = (x \cap a_1) \cup \cdots \cup (x \cap a_n)$. If each $a_j$ is an atom of $B_1$, each $x \cap a_j$ is 0 or $a_j$, so $x \in B$. $\therefore B = B_1$, contradiction. ∎

DEFINITION. Suppose $B$ is a subalgebra of $B_1$, and $x \in B_1$. Let $\langle B, x\rangle$ be the subalgebra of $B_1$ generated by $B$ and $x$.

LEMMA 13. *Let $B_1, B_2$ be finite Boolean algebras, and $f: B_1 \cong B_2$ an isomorphism. Let $B_1' = \langle B_1, x\rangle$, and $B_2' = \langle B_2, y\rangle$ be extensions of $B_1, B_2$, respectively. Suppose $0 < x < a$, where $a$ is an atom of $B_1$, and $0 < y < f(a)$. Then $f$ extends to a unique isomorphism $g: B_1' \cong B_2'$ with $g(x) = y$.*

*Proof.* Let $a = a_0, a_1, \ldots, a_n$ be the atoms of $B_1$. Let $b_i = f(a_i)$, $i \leqslant n$. Then the $b_i$ are the atoms of $B_2$. For $i \geqslant 1$, $x \cap a_i \leqslant a_0 \cap a_i = 0$, so $x \cap a_i = 0$. Similarly $y \cap b_i = 0$.

Let $t \in B_1$. Then $t = t \cap 1 = (t \cap a_0) \cup \cdots \cup (t \cap a_n)$, so

$$\begin{aligned} x \cap t &= x \cap t \cap a_0 \\ &= 0 \qquad \text{if} \quad t \cap a_0 = 0; \\ &= x \qquad \text{if} \quad t \cap a_0 = a_0. \end{aligned}$$

Similarly,

$$\begin{aligned} y \cap f(t) &= 0 \qquad \text{if} \quad f(t) \cap b_0 = 0; \\ &= y \qquad \text{if} \quad f(t) \cap b_0 = b_0. \end{aligned}$$

Now we switch to the ring formulation.

Every element of $B_1'$ is uniquely of the form $\beta_1 + \lambda_1 \cdot x$, $\beta_1 \in B_1$, $\lambda_1 = 0$ or 1. Similarly for $B_2'$. We have calculated $x \cdot \beta_1$ for each $\beta_1 \in B_1$, and since $x^2 = x$ the multiplication table of $B_1'$ is uniquely determined. Similarly, the multiplication table for $B_2'$ is uniquely determined, and $f$ extends to a unique Boolean-ring isomorphism $g: B_1' \cong B_2'$ with $g(x) = y$. ∎

Now we extend Lemma 13 a little, to enable us to handle isomorphisms between augmented Boolean algebras. First we need a definition.

DEFINITION. Suppose $f: B_1 \cong B_2$ is an isomorphism of Boolean algebras. Let $I_1$ be an ideal in $B_1$, and $I_2$ an ideal in $B_2$. Then, $f$ is an $(I_1, I_2)$-map if $f(I_1) = I_2$.

We shall use the notation $x \equiv a \pmod I$ to mean $x - a \in I$.

LEMMA 14. *Let $B_1, B_2$ be finite Boolean algebras with ideals $I_1, I_2$, respectively. Let $B_1' = \langle B_1, x\rangle$, $B_2' = \langle B_2, y\rangle$, be extensions of $B_1, B_2$, respectively, with ideals $I_1', I_2'$, respectively, such that $I_1' \cap B_1 = I_1$, $I_2' \cap B_2 = I_2$. Suppose $f: B_1 \cong B_2$ is an $(I_1, I_2)$-map. Suppose $0 < x < a$, where $a$ is an atom of $B_1$, and $0 < y < f(a)$.*

*Suppose that one of the following conditions holds.*

(1) $x \in I_1'$ *and* $y \in I_2'$;

(2) $x \equiv a \pmod{I_1'}$, *and* $y \equiv f(a) \pmod{I_2'}$;

(3) $x \notin I_1'$, $x \not\equiv a \pmod{I_1'}$, $y \notin I_2'$, *and* $y \not\equiv f(a) \pmod{I_2'}$.

*Then $f$ extends to a unique $(I_1', I_2')$-map $g: B_1' \cong B_2'$ with $g(x) = y$.*

*Proof.* Clearly we have only to verify that the $g$ given by Lemma 13 is an $(I_1', I_2')$-map.

Every element of $B_1'$ is uniquely of the form $\beta_1 + \lambda_1 \cdot x$, where $\beta_1 \in B_1$ and $\lambda_1 = 0$ or 1. We have $g(\beta_1 + \lambda_1 \cdot x) = f(\beta_1) + \lambda_1 \cdot y$.

Assume (1). Then $\beta_1 + \lambda_1 \cdot x \equiv \beta_1 \pmod{I_1'}$, so $\beta_1 + \lambda_1 \cdot x \in I_1'$ if and only if $\beta_1 \in I_1'$. But $I_1' \cap B_1 = I_1$, so $\beta_1 + \lambda_1 \cdot x \in I_1'$ if and only $\beta_1 \in I_1$. Similarly, $f(\beta_1) + \lambda_1 \cdot y \in I_2'$ if and only if $f(\beta_1) \in I_2$. Since $f$ is an $(I_1, I_2)$-map, it follows that $g$ is an $(I_1', I_2')$-map, as required.

Assume (2). Then, essentially as in the proof of (1), we can show that $g$ is an $(I_1', I_2')$-map.

Assume (3). Suppose $\beta_1 + x \in I_1'$. Then

$$x \equiv -\beta_1 \pmod{I_1'},$$

so

$$x \equiv \beta_1 \pmod{I_1'},$$

so

$$x = x \cap a \equiv \beta_1 \cap a \pmod{I_1'}.$$

Since $\beta_1 \cap a = a$ or $\beta_1 \cap a = 0$, we get $x \equiv 0 \pmod{I_1'}$, or $x \equiv a \pmod{I_1'}$. Either conclusion gives a contradiction. Thus $\beta_1 + x \notin I_1'$. Similarly $f(\beta_1) + y \notin I_2'$. Thus $g$ is an $(I_1', I_2')$-map. ▮

The preceding lemmas will be our main tools. We give one more lemma of the same type, designed to show that the hypotheses of the preceding lemmas are natural.

LEMMA 15. *Let $B$ be a finite Boolean algebra with an ideal $I$. Let $B'$ be a proper extension of $B$, of the form $\langle B, y\rangle$. Let $I'$ be an ideal of $B'$, with $I' \cap B = I$. Then there is an atom $a$ of $B$, and an element $x$ of $B'$ such that $0 < x < a$ and $B' = \langle B, x\rangle$. Moreover, either*

(1) *$x \in I'$, or*

(2) *$x - a \in I'$, or*

(3) *for all $b$ in $B$, $x - b \in I'$.*

*Proof.* Suppose $B$ has $n$ atoms. Then $B$ has cardinality $2^n$. Then clearly, $B'$ has cardinality $2^{n+1}$.

By Lemma 12, we can select an atom $a$ of $B$ such that $a$ is not an atom of $B'$. Select $x$ in $B'$ with $0 < x < a$. Then clearly $\langle B, x\rangle$ has cardinality $2^{n+1}$, so $B' = \langle B, x\rangle$.

Suppose $b \in B$, and $x - b \in I'$. Then $x = x \cap a \equiv b \cap a \pmod{I'}$. But $b \cap a = 0$ or $a$. Therefore $x \equiv 0 \pmod{I'}$ or $x \equiv a \pmod{I'}$. This proves the lemma. ▮

We have now assembled the apparatus for extending isomorphisms. The problem is to get them started.

3.3.2. *Frames. For the remainder of this section we restrict our attention to augmented Boolean algebras $\mathcal{M}$ such that $H_0(\mathcal{M})$ is finite, and for each $J$ in $H_0(\mathcal{M})$, $\mathcal{M}_0/J$ has only finitely many atoms.*

For $J \in H_0(\mathcal{M})$, let $\eta_J$ be the canonical quotient map $\mathcal{M}_0 \to \mathcal{M}_0/J$.

DEFINITION. An element $a$ of $\mathcal{M}$ is a *$J$-atom* if $\eta_J(a)$ is an atom of $\mathcal{M}_0/J$.

DEFINITION. A *frame* for $\mathcal{M}$ is a Boolean algebra $B$ such that

(i) $B \subseteq \mathcal{M}_0$;

(ii) for each $J \in H_0(\mathcal{M})$, and each $J$-atom $a$, there is a $b$ in $B$ such that $a \equiv b \pmod{J}$.

LEMMA 16. *$\mathcal{M}$ has a finite frame.*

*Proof.* Since $\mathscr{M}_0/J$ has only finitely many atoms, for each $J$ in $H_0(\mathscr{M})$, we can select for each $J$ a finite set $\mathscr{A}_J$ such that each $J$-atom is congruent modulo $J$ to an element of $\mathscr{A}_J$. Let $B$ be the algebra generated by the union of the sets $\mathscr{A}_J$. Since $H_0(\mathscr{M})$ is finite, $B$ is finitely generated, and so finite. $B$ is clearly a frame for $\mathscr{M}$. ∎

3.3.3. Suppose $\mathscr{M}^{(1)}$ and $\mathscr{M}^{(2)}$ are augmented Boolean algebras $(B^{(1)}, J_i^{(1)}, i < n)$ and $(B^{(2)}, J_i^{(2)}, i < n)$, satisfying the conditions set down at the beginning of 3.3.2.

*Suppose in this subsection that* $\mathscr{M}^{(1)} \equiv \mathscr{M}^{(2)}$. We will deduce certain conditions on $\mathscr{M}^{(1)}$ and $\mathscr{M}^{(2)}$, and in due course prove that these conditions imply that $\mathscr{M}^{(1)} \equiv_{\infty,\omega} \mathscr{M}^{(2)}$.

CONDITION 1. *There is a Heyting algebra isomorphism* $\psi$: $H_0(\mathscr{M}^{(1)}) \cong H_0(\mathscr{M}^{(2)})$ *such that* $\psi(J_i^{(1)}) = J_i^{(2)}$, $i < n$.

To see this, recall that the Heyting algebra operations $\wedge$, $\vee$, and $\rightarrow$, are first-order definable in $\mathscr{M}^{(1)}$ and $\mathscr{M}^{(2)}$. Condition 1 now follows since $\mathscr{M}^{(1)} \equiv \mathscr{M}^{(2)}$. ∎

We note also that the map $\psi$ is unique, and henceforward we reserve the notation "$\psi$" for this particular map.

CONDITION 2. *For each* $J$ *in* $H_0(\mathscr{M}^{(1)})$,

(a) *either* $B^{(1)}/J$ *and* $B^{(2)}/\psi(J)$ *have the same finite cardinality, or both are infinite*;

(b) $B^{(1)}/J$ *and* $B^{(2)}/\psi(J)$ *have the same number of atoms.*

This is clear. ∎

CONDITION 3. *There is a finite frame* $\mathscr{A}^{(1)}$ *of* $\mathscr{M}^{(1)}$, *and a (Boolean algebra) isomorphism* $\Phi$ *of* $\mathscr{A}^{(1)}$ *into* $B^{(2)}$ *such that* $\Phi(\mathscr{A}^{(1)})$ *is a frame for* $B^{(2)}$, *and* $\Phi(\mathscr{A}^{(1)} \cap J) = \Phi(\mathscr{A}^{(2)}) \cap \psi(J)$ *for each* $J$ *in* $H_0(\mathscr{M})$.

To prove this, we first note that $\mathscr{M}^{(1)}$ has a finite frame $\mathscr{A}^{(1)}$, by Lemma 16. Let $x_0, \dots, x_k$ be the elements of $\mathscr{A}^{(1)}$. It is an easy exercise to construct a first-order sentence $S$ which expresses the following:

The Heyting algebra $H$ generated by $J_i$, $i < n$, is isomorphic to $H_0(\mathscr{M})$ by the correspondence $\tau$ sending $J_i$ to $J_i^{(1)}$. There are elements $v_0, \dots, v_k$ which form a frame, such that the augmented Boolean algebra consisting of $v_0, \dots, v_k$, with the ideals $\{v_0, \dots, v_k\} \cap J$ for $J \in H$, is isomorphic to the augmented algebra consisting of $x_0, \dots, x_k$, with the ideals $\{x_0, \dots, x_k\} \cap \tau(J)$ for $J \in H$.

Then $\mathscr{M}^{(1)} \models S$, so $\mathscr{M}^{(2)} \models S$, whence Condition 3 follows easily. ∎

3.3.4. In this subsection we suppose that $\mathscr{M}^{(1)}$ and $\mathscr{M}^{(2)}$ satisfy Conditions 1–3. We fix a frame $\mathscr{A}^{(1)}$ and a map $\Phi$ as in Condition 3, and let $\mathscr{A}^{(2)} = \Phi(\mathscr{A}^{(1)})$.

Note the symmetry of the situation. If we interchange $\mathscr{M}^{(1)}$ and $\mathscr{M}^{(2)}$, $\mathscr{A}^{(1)}$ and $\mathscr{A}^{(2)}$, and replace $\psi$, $\Phi$ by $\psi^{-1}$, $\Phi^{-1}$, respectively, then we are again in the situation of the previous paragraph.

We wish to prove that $\mathscr{M}^{(1)} \equiv_{\infty,\omega} \mathscr{M}^{(2)}$. We will show that Karp's criterion [11] applies.

Let $\mathscr{K}$ be the set of maps $f$ such that

(i) $f$ is an isomorphism of a finite subalgebra of $B^{(1)}$ to a finite subalgebra of $B^{(2)}$;

(ii) $f$ extends $\Phi$;

(iii) if $y \in \mathrm{dom}(f)$ and $J \in H_0(\mathscr{M}^{(1)})$, then $y \in J \Leftrightarrow f(y) \in \psi(J)$.

We claim that $\mathscr{K}$ satisfies the conditions of Karp's criterion. By Condition 3, $\mathscr{K} \neq \varnothing$. Because of the symmetry of the situation, to verify that Karp's criteria are met, we need only prove: If $f \in \mathscr{K}$, and $t \in B^{(1)}$, there exists $g \in \mathscr{K}$, with $f \subseteq g$ and $t \in \mathrm{dom}(g)$.

So, suppose $f \in \mathscr{K}$, and $t \in B^{(1)}$. Let $D^{(1)} = \mathrm{dom}(f)$. If $t \in D^{(1)}$ there is nothing to prove. So suppose $t \notin D^{(1)}$. By Lemma 15, there is an atom $a$ of $B^{(1)}$, and an element $x \in B^{(1)}$ with $0 < x < a$, such that $\langle D^{(1)}, t\rangle = \langle D^{(1)}, x\rangle$. Also, for each $J \in H_0(\mathscr{M})$, either $x \in J$ or $x - a \in J$, or $x - b \notin J$ for all $b \in D^{(1)}$.

By the proof of Lemma 14, if we can find $y$ in $B^{(2)}$ with $0 < y < f(a)$, and such that, for all $J \in H_0(\mathscr{M}^{(1)})$, $x \in J \Leftrightarrow y \in \psi(J)$ and $x - a \in J \Leftrightarrow y - f(a) \in \psi(J)$, then $f$ extends to an isomorphism $g$ of $\langle D^{(1)}, x\rangle$ into $B^{(2)}$, and $g$ will be an element of $\mathscr{K}$.

We have at last isolated the main difficulty. In the next subsection, we show that we have arranged matters so that this difficulty can be oversome.

3.3.5. (Notation as in 3.3.4). Let $f$, $a$, $x$ be as above. We now prove that there exists $y$ in $B^{(2)}$ meeting the above requirements.

Let $X$, $Y$, $Z$ be, respectively, the sets of $J \in H_0(\mathscr{M}^{(1)})$ such that

(i) $x \in J$,

(ii) $x - a \in J$,

(iii) $x \notin J$ and $x - a \notin J$.

Neither $X$ nor $Y$ is empty, since the ideal $B^{(1)}$ belongs to them.

THEOREM 10. *The only totally atomless $\aleph_0$-categorical augmented Boolean structures $\langle B, I\rangle$ are those where the Heyting algebra generated by $I$ is isomorphic to*

$$D_{p2_q}, \quad D_{p2}, \quad D_{pq}, \quad \text{or} \quad D_p.$$

This gives an explicit description of all totally atomless $\aleph_0$-categorical structures of form $\langle B, I\rangle$, where $I$ is an ideal of $B$.

## REFERENCES


1. R. F. ARENS AND I. KAPLANSKY, Topological representation of algebras, *Trans. Amer. Math. Soc.* **63** (1948), 457–481.
2. C. J. ASH, $\aleph_0$-categorical theories, to appear.
3. G. BIRKHOFF, "Lattice Theory," American Mathematical Society Colloquium Publications, Vol. 25, American Mathematical Society, Providence, R.I., 1940.
4. C. C. CHANG, Omitting types of prenex formulas, *J. Symbolic Logic* **32** (1967), 61–73.
5. S. D. COMER, Elementary properties of structures of sections, to appear.
6. W. GLASSMIRE, JR., There are $2^{\aleph_0}$ countably categorical theories, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **19** (1971), 185–190.
7. A. GRZEGORCZYK, Logical uniformity by decomposition and categoricity in $\aleph_2$, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **16** (1968), pp. 687–692.
8. P. R. HALMOS, "Lectures on Boolean Algebras," Van Nostrand, Princeton, N.J., 1963.
9. I. N. HERSTEIN, "Noncommutative Rings," Carus Mathematical Monographs, No. 15, Mathematical Association of America, Washington, D.C., 1968.
10. N. JACOBSON, "Structure of Rings," American Mathematical Society Colloquium Publications, Vol. 37, American Mathematical Society, Providence, R. I., 1964.
11. C. KARP, Finite-quantifier equivalence, *in* "The Theory of Models, pp. 407–412, Proceedings of the 1963 International Symposium at Berkely, North-Holland, Amsterdam, 1965.
12. A. MACINTYRE, Categoricity in power for some algebraic theories, *J. Symbolic Logic* **35** (1970), 606.
13. A. MACINTYRE, Model-completeness for sheaves of structures, *Fund. Math.* **81** (1973), 73–89.
14. N. H. McCOY, Subrings of direct sums, *Amer. J. Math.* **60** (1938), 374–382.
15. I. NISHIMURA, On Formulas of one variable in intuitionistic propositional calculus, *J. Symbolic Logic* **25** (1960), 327–331.
16. M. O. RABIN, Decidability of second-order theories and automata on infinite trees, *Trans. Amer. Math. Soc.* **141** (1969), 1–34.
17. J. G. ROSENSTEIN, $\aleph_0$-categoricity of linear orderings, *Fund. Math.* **64** (1969), 1–5.
18. J. G. ROSENSTEIN, $\aleph_0$-categoricity of Groups, *J. Algebra* **25** (1973), 435–467.
19. C. RYLL-NARDZEWSKI, On the categoricity in power $\leqslant \aleph_0$, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom Phys.* **7** (1959), 545–548.
20. G. SACKS, "Saturated Model Theory," Benjamin, New York, 1972.
21. D. SARACINO, Model companions for $\aleph_0$-categorical theories, *Proc. Amer. Math. Soc.*, **39** (1973), 591–598.
22. J. WASKIEWICZ AND B. WEGLORZ, On $\omega_0$-categoricity of powers, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* **17** (1969), 195–199.


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