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EFFECTIVE MATCHMAKING AND k -CHROMATIC GRAPHS

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ABSTRACT. In an earlier paper we showed that there is a recursive society, in which each person knows exactly two other people, whose marriage problem is solvable but not recursively solvable. We generalize this result, using a different construction, to the case where each person knows exactly k other people. From this we deduce that for each $k \geq 2$ there is a recursive $2(k-1)$ -regular graph, whose chromatic number is k but which is not recursively k -chromatic.

1. Graphs, societies, and algorithms. Following Berge [1] a set S of unordered pairs of distinct elements of a set P determines a graph $\Gamma = (P, S)$. The elements of P are called *points* or *vertices* of Γ ; the elements of S are called *arcs* of Γ . It is not assumed that P or S is finite. Points x and y are said to be *adjacent* if $\{x, y\}$ is an arc. Γ is *k -chromatic* if the points of Γ can be painted with k colors in such a way that no two adjacent points are of the same color. The *chromatic number* of Γ is the smallest number k such that Γ is k -chromatic. Γ is *k -regular* if every point of Γ is adjacent to exactly k points. Γ is called *simple* (or *bipartite*) if there exist disjoint sets B and G such that $P = B \cup G$ and if wherever $\{x, y\} \in S$ then $x \in B$ and $y \in G$ or $y \in B$ and $x \in G$. Two distinct arcs are said to be *adjacent* if they have a point in common. A *matching* of a simple graph (B, G, S) is a set W of arcs no two of which are adjacent. Let W be a matching, $B_{1W} = \{b \in B \mid \text{for some } g, \{b, g\} \in W\}$, and $G_{1W} = \{g \in G \mid \text{for some } b, \{b, g\} \in W\}$; W is then said to be a *matching of B_{1W} onto G_{1W}* or a *matching of B_{1W} into G* .

We now recall the more colorful, anthropomorphic terminology of Halmos and Vaughan [3]. Let $\Sigma = (B, G, S)$ be a simple graph. We call

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Σ a society, B the set of boys, and G the set of girls. If b and g are adjacent in the graph Σ we say that b and g are *acquainted* in the society Σ . We call the society Σ a *k-society* if as a graph it is *k-regular*, so that in a *k-society* each person knows exactly *k* people of the opposite sex. The society Σ is said to have a *solvable marriage problem* if there is a matching of B into G , for we can think of the matching as providing, in a monogamous way, a mate for each boy from among the girls he knows. Similarly, Σ is said to have a *symmetric solution to its marriage problem* if there is a matching of B onto G .

We also associate with the society Σ another graph Γ_{Σ} as follows. The points of Γ_{Σ} are the arcs of Σ and the arcs of Γ_{Σ} are the unordered pairs of adjacent arcs of Σ . If Σ is a *k-society* then Γ_{Σ} is a $2(k-1)$ -regular graph.

We use three combinatorial lemmas which we state here without proof.

LEMMA 1. *If Σ is a k-society, then there is a symmetric solution to its marriage problem.*

LEMMA 2. *If Σ is a k-society, the chromatic number of Γ_{Σ} is k.*

LEMMA 3. *Let Σ be a k-society. The set of points of Γ_{Σ} possessing a common color in a k-coloring of Γ_{Σ} is a matching of B onto G in Σ . Thus such a set of points is a solution to the symmetric marriage problem of Σ .*

In the finite case, Lemmas 1 and 2 are just restatements of results due to König and P. Hall appearing in Berge [1, pp. 92-95]. To prove these lemmas in the infinite case, one can, for example, use O. Ore's extension of the Schroeder-Bernstein theorem (Theorem 1.3.4 in Mirsky [4]). Lemma 3 is easily verified directly.

Following Rogers [5] a function which is computable by an algorithm or an effective procedure is called a *partial recursive function*. The domain of a partial recursive function is assumed to be a subset of N^m for a fixed m (N is the set of natural numbers) and its range is assumed to be a subset of N . If its domain happens to be all of N^m the partial recursive function θ is called a (*general*) *recursive function*. If x is in the domain of θ we say that $\theta(x)$ is defined; otherwise we say that $\theta(x)$ is undefined. A set is said to be *recursive* if its characteristic function is a recursive function.

The collection of all finite sets of instructions, or algorithms, formulated in a fixed language can be recursively (i.e. effectively) enumerated. Assuming this to be done, ϕ_e denotes the partial recursive function defined by the *e*th finite set of instructions. Given an argument x and a number e of a set of instructions, it is not possible to determine effectively whether or not $\phi_e(x)$ is defined. However, it is possible, for each n , to determine effectively—in e , x , and n —whether or not $\phi_e(x)$ is defined in n steps by simply carrying out n steps of the *e*th algorithm applied to x and

observing the outcome. " $\phi_e^n(x)$ is defined" will mean that $\phi_e(x)$ is defined in n steps; in that case the value of $\phi_e^n(x)$ will be $\phi_e(x)$. Furthermore if $\phi_e(x)$ is defined there must be an n such that $\phi_e^n(x)$ is defined—and for all $n' \geq n$, $\phi_e^{n'}(x)$ is defined and equals $\phi_e(x)$. The formal statements and verifications of these remarks can be found, for example, in Rogers [5, Theorems 1-IV and 1-IX].

In what follows a society will also satisfy the conditions (i) each person knows only finitely many other people (i.e. Σ is locally finite) and (ii) everyone knows someone. If all but (ii) are satisfied by Σ , then Σ will be called a *partial society*. The connected components of a partial society Σ are called the *communities of Σ* .

We say that the society Σ is *recursive* if B is the set of even natural numbers, G is the set of odd natural numbers, and S is a recursive set of ordered pairs. We will use $B(n)$ for $2n$ and $G(n)$ for $2n+1$ and say that $B(n)$ is the n th boy and that $G(n)$ is the n th girl. A recursive society is said to be *recursively (symmetrically) solvable* if there is a 1-1 (onto) recursive function f such that, for each n , $B(n)$ knows $G(f(n))$.

We say that the graph $\Gamma = (P, S)$ is *recursive* if $P = N$ and S is a recursive set of ordered pairs. Γ is said to be *recursively k -chromatic* if there is a recursive function f of one variable whose range is a subset of $\{0, 1, \dots, k-1\}$ such that if x is adjacent to y then $f(x) \neq f(y)$.

Let Σ be a recursive society and let j be a 1-1 function which maps S recursively onto N . Define $j(\Gamma_\Sigma)$ to be the graph whose points are N and whose arcs are the pairs $\{j(b, g), j(b', g')\}$ such that $\{(b, g), (b', g')\}$ is an arc of Γ_Σ . Observe that if Σ is a recursive society, $j(\Gamma_\Sigma)$ is a recursive graph. Since $j(\Gamma_\Sigma)$ is isomorphic to Γ_Σ , we know that if Σ is a k -society, $j(\Gamma_\Sigma)$ is a $2(k-1)$ -regular graph, and that, by Lemma 2, $j(\Gamma_\Sigma)$ has chromatic number k . Lemma 3 shows that if $j(\Gamma_\Sigma)$ is recursively k -chromatic, then Σ is recursively solvable. These observations show that the following corollary is a consequence of the existence of a recursive k -society which is not recursively solvable. This will be proved in §2.

COROLLARY. *There is a recursive $2(k-1)$ -regular graph whose chromatic number is k , but which is not recursively k -chromatic.*

2. Recursive k -societies without recursive solutions.

THEOREM. *For each $k \geq 2$ there is a recursive k -society which is symmetrically solvable but is not recursively solvable.*

PROOF. In the proof we construct a recursive society Σ by stages; at stage n , for each $n > 0$, a partial society $\Sigma_n = (B, G, S_n)$, with S_n finite, is effectively defined so that, for each n , $S_n \subseteq S_{n+1}$ and so that $\Sigma =$

into S_n ” we will say “introduce x to y ” or “introduce y to x ” at stage n . A person is called a “stranger” at a given point in the construction if he has not yet been introduced to anyone. At the beginning of each stage n of the construction there are numbers a and b (with $a \geq n$ and $b \geq n$) such that the first a boys and b girls are not strangers at that point, but the remaining boys and girls are; we will reserve the numbers a and b for this purpose, so that $B(a)$ and $G(b)$ always are the first male and female strangers at the beginning of the appropriate stage of the construction.

For each n , each introduction made during stage n involves at least one person who was a stranger at the beginning of stage n . This feature, together with the effectiveness of the construction of S_n , implies that Σ is recursive. To see this we show how to decide whether or not $B(p)$ knows $G(q)$. Let $n > p$ and $n > q$. Since the first male and female strangers at stage n are $B(a)$ and $G(b)$ where $a \geq n$ and $b \geq n$ it follows that $B(p)$ and $G(q)$ have acquaintances in Σ_n . Hence $B(p)$ knows $G(q)$ in Σ if and only if he already knows her in Σ_n . But whether or not he knows her in Σ_n can be effectively determined by effectively reconstructing S_n .

The community of the partial society Σ_{n-1} to which $B(i)$ belongs at the beginning of stage n will be denoted $C_n(i)$. $C_n(i)$ is called *stable* if $C_m(i) = C_n(i)$ for all $m \geq n$. The remarks in the preceding paragraph show that if $C_n(i)$ is stable, then no member of $C_n(i)$ will ever meet someone new. In particular, if $C_n(i)$ is stable and $B(p)$ and $G(q)$ are in $C_n(i)$ but cannot marry in $C_n(i)$ (i.e. there is no solution to the marriage problem of $C_n(i)$ in which $B(p)$ marries $G(q)$), then $B(p)$ cannot marry $G(q)$ in Σ .

We now define simultaneously the recursive society Σ and k 1-1 recursive functions r_0, r_1, \dots, r_{k-1} (with pairwise disjoint ranges); at the end of stage n , $r_t(i)$ will be defined for all $i < n$ and $t < k$.

Intuitively, the construction will guarantee that if $\phi_e(r_t(e))$ is defined for all $t < k$ than no solution to the marriage problem of Σ simultaneously marries each $B(r_t(e))$ to the corresponding $G(\phi_e(r_t(e)))$, so that ϕ_e cannot be a solution to the marriage problem of Σ . Since every recursive function is ϕ_e for some e , this implies that the marriage problem of Σ has no recursive solution.

We assume as part of the induction hypothesis that at stage n for each $i < n$ either all $B(r_t(i))$ are in the same community or they are in k different communities. In the former case, the community is a stable one in which each person knows exactly k others. In the latter case there is a p such that for each $t < k$ the community $C_n(B(r_t(i)))$ contains exactly $1 + (k-1)k + (k-1)^3k + \dots + (k-1)^{2p+1}k$ boys and $k + (k-1)^2k + \dots + (k-1)^{2p}k$ girls, and can be put into 1-1 correspondence with the nodes of the graph g_p below in such a way that boys correspond to nodes marked $\♂$, girls

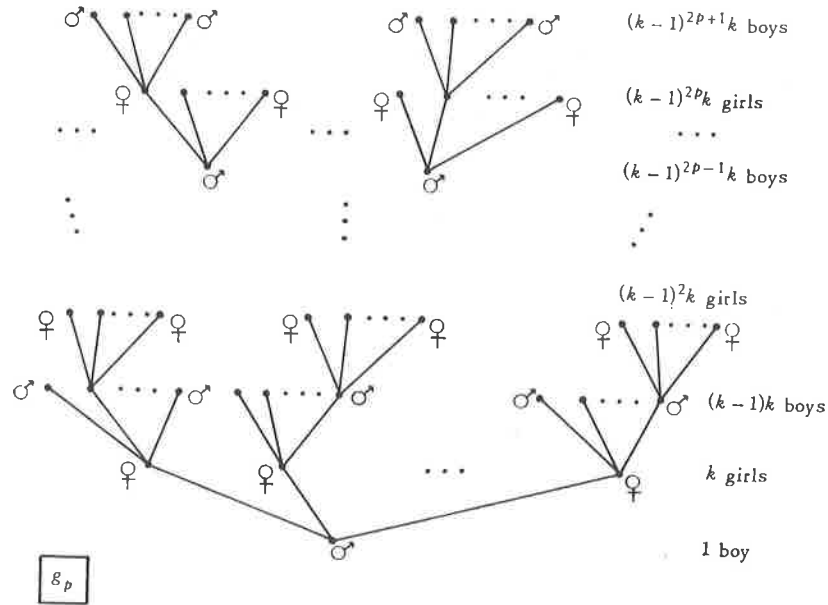


FIGURE 1

and two nodes are adjacent if and only if the people mapped to these nodes know each other. If this is the case we shall say that $C_n(B(r_i(i)))$ has form g_p . We assume that the definition of the j th row of g_p , for $0 \leq j \leq 2p+2$, and of the i th position (from the left) on the j th row of g_p , for $0 \leq i < (k-1)^{j-1}k$ where $j \geq 1$, need not be made explicit. It is also clear what we mean when we say that (under a particular correspondence) a certain person of the community C (which has form g_p) is in the i th position of the j th row of C . Note that in a community of form g_p each person except those on the top, i.e. $(2p+2)$ th, row know exactly k other people.

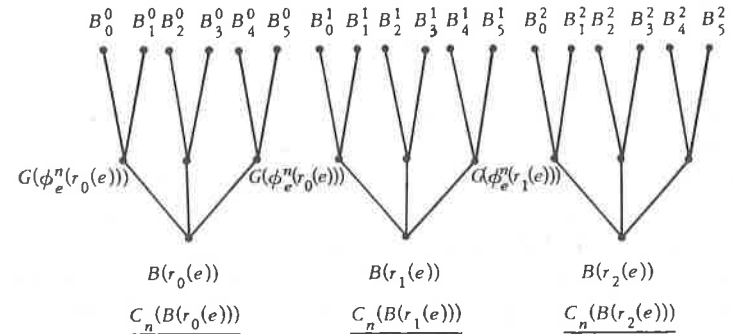
Stage $n > 0$. Define $r_t(n-1) = a+t$ for each $t < k$ (the first k unused boys) and establish for each $B(r_t(n-1))$ a community containing k additional new girls, and $k(k-1)$ additional new boys, so that it has the form g_0 .

Let $n = 2^e q$ where q is odd, say $q = 2m+1$. If all $B(r_t(e))$ are already in the same community proceed to stage $n+1$. If they are still in different communities, and if either some $\phi_e^n(r_t(e))$ is undefined or all are defined but some $B(r_t(e))$ does not know $G(\phi_e^n(r_t(e)))$, then, since each of the k communities is of the form g_p where $p = m-1$, we introduce each of the $(k-1)^{2p+1}k$ boys in the top row of each community to $k-1$ new girls and each of these $(k-1) \cdot (k-1)^{2p+1}k$ girls to $(k-1)$ new boys, so that

the resulting k communities are all of the form g_{p+1} . Finally, we consider the case where the $B(r_t(e))$ are in different communities, where all $\phi_e^n(r_t(e))$ are defined, and where $B(r_t(e))$ knows $G(\phi_e^n(r_t(e)))$ for each $t < k$. We assume that each $C_n(B(r_t(e)))$ has the form g_p and that the correspondence between $C_n(B(r_t(e)))$ and the nodes of g_p places $G(\phi_e^n(r_t(e)))$ in the left-most, i.e. 0th, position in the first row of $C_n(B(r_t(e)))$, for each $t < k$. [At most a relabelling is necessary to guarantee this.] Let B_i^t be the boy in the i th position of the top row of $C_n(B(r_t(e)))$ for each $i < T = (k-1)2^{p+1}k$ and each $t < k$. Let $G_0, G_1, \dots, G_{T(k-1)-1}$ denote the first $T(k-1)$ female strangers. Introduce B_i^t to each of $G_i, G_{T+i}, G_{2T+i}, \dots, G_{(k-2)T+i}$ for each $t < k$ and each $i < T$.

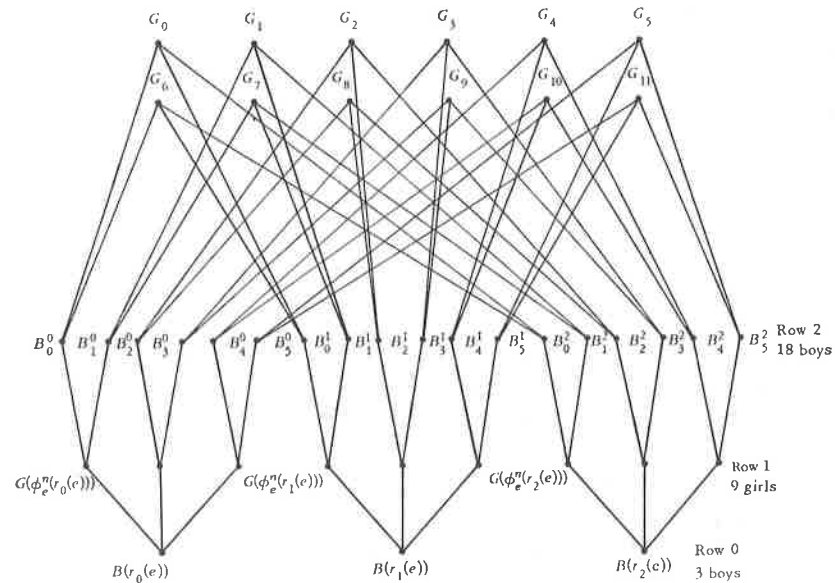
This completes the construction of Σ . Before we proceed to prove that it satisfies the desired properties we shall consider the following situation which contains within it the essence of the argument.

Suppose then that $k=3$ and that e is such that at stage $n=2^p$ we have that $\phi_e^n(r_t(e))$ is defined for each $t < 3$ and that $B(r_t(e))$ knows $G(\phi_e^n(r_t(e)))$ for each $t < 3$. At this stage each $C_n(B(r_t(e)))$ has the form g_0 . After rearrangement these communities take the form



Thus the final case of the construction is the relevant one. After it is applied, we obtain the community $C_{n+1}(B(r_0(e)))$ which assumes the form below. It is now evident that in no solution to the marriage problem of $C_{n+1}(B(r_0(e)))$ can $B(r_t(e))$ marry $G(\phi_e^n(r_t(e)))$ for each $t < 3$. For, of B_0^0, B_1^0, B_2^0 exactly two marry G_0 and G_6 ; the remaining one B_3^0 must marry $G(\phi_e^n(r_1(e)))$ so that $B(r_1(e))$ cannot marry her. Similarly, of B_0^1, B_1^1, B_2^1 exactly two marry G_1 and G_7 ; the remaining one B_3^1 must marry $G(\phi_e^n(r_2(e)))$ so that $B(r_2(e))$ cannot marry her. Hence, in fact, only (and exactly) one of $B(r_t(e))$ marries $G(\phi_e^n(r_t(e)))$.

We return now to the general case. It is clear from the construction that Σ is a recursive society (note that each introduction involves a stranger)



in it knows exactly k others, or has form g where g is the direct limit of the graphs g_p ; so again each person in it knows exactly k other people. That Σ is symmetrically solvable follows from Lemma 1.

Thus we need only show that Σ is not recursively solvable—i.e. that no ϕ_e is a solution to the marriage problem of Σ . It suffices, of course, to show that if $\phi_e(r_t(e))$ is defined for each $t < k$ and $B(r_t(e))$ knows $G(\phi_e(r_t(e)))$ for each $t < k$ then no solution to the marriage problem of $C(B(r_0(e)))$ marries each $B(r_t(e))$ to the corresponding $G(\phi_e(r_t(e)))$. Note of course that under these hypotheses at some stage n we combine the $C_n(B(r_t(e)))$ into one community which is stable at stage n . We may assume that at stage n each $C_n(B(r_t(e)))$ has the form g_p (for some p) and that $G(\phi_e(r_t(e)))$ is in the leftmost position in the first row of $C_n(B(r_t(e)))$.

We shall show, by induction on $j < 2p + 3$ that for each i if A_t is the person in the i th position of the $(2p + 3 - j)$ th row of $C_n(B(r_t(e)))$ for $t < k$ then in any solution to the marriage problem of $C_n(B(r_t(e)))$ exactly one of $\{A_t | t < k\}$ marries a person on the row below—i.e. on the $(2p + 2 - j)$ th row. Thus taking $j = 2p + 2$ and $i = 0$ we conclude that exactly one of $G(\phi_e(r_t(e)))$ marries $B(r_t(e))$, hence certainly not every $G(\phi_e(r_t(e)))$ marries $B(r_t(e))$.

For $j = 0$ we must consider, for each fixed i , the boys $B_i^0, B_i^1, \dots, B_i^{k-1}$ on the top row. Now at stage n each of these k boys was introduced to the $k - 1$ girls $G_i, G_{T+i}, G_{2T+i}, \dots, G_{(k-2)T+i}$. In any fixed solution to the marriage problem some B_i^t must marry a girl other than these; but the

only other girl he knows is on the row below him. Also, since we added $[(k-1)^{2p+1}k] \cdot (k-1)$ girls at stage n , the total number of girls in $C_{n+1}(B(r_0(e)))$ is $[k+(k-1)^2k+\cdots+(k-1)^{2p}k]k+(k-1)^{2p+2}k$ which equals the total number of boys

$$[1+(k-1)k+(k-1)^3k+\cdots+(k-1)^{2p+1}k]k$$

in $C_{n+1}(B(r_0(e)))$, so that any solution to the marriage problem of $C_{n+1}(B(r_0(e)))$ is symmetric. Hence each G_{pT+i} must marry one of B_i^t . Hence exactly one B_i^t marries a girl on the row below him.

Now assume that the claim is proven for $j < 2p+3$ and suppose that $j+1 < 2p+3$. Then each A_i in the i th position of the $(2p+2-j)$ th row knows exactly $k-1$ people on the row above the $(2p+3-j)$ th row, and these people are in the $(i(k-1))$ th, $(i(k-1)+1)$ th, \cdots , $(i(k-1)+(k-2))$ th positions on the $(2p+3-j)$ th row. Now exactly one of the people in the $(i(k-1)+s)$ th position marries a person below him, for each $s < k-1$. Thus exactly $k-1$ of $A_0, A_1, \cdots, A_{k-1}$ marry people in the row above them. Hence exactly one of them marries a person in the row below. This completes the induction and the proof. \square

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