Mathematics Standards, Division, and Constructivism

by Joseph G. Rosenstein

The day after "mathematics education" was featured on the New York Times Op Ed page [in the summer of 1997], I was scheduled to speak to a group of high school mathematics teachers and mathematics and computer science researchers about "The Standards Approach to Education". This took place at the DIMACS Research and Education Institute (DREI) at Rutgers University. The program and its sponsoring organization -- the Center for Discrete Mathematics and Theoretical Computer Science (DIMACS) -- are funded by the National Science Foundation.

My choice of topic reflected my background as professor of mathematics at Rutgers University and as Director of the New Jersey Mathematics Coalition, as a result of which I played a major role in developing and promoting New Jersey's recently [i.e., a year ago] adopted mathematics standards and accompanying curriculum framework.

The headline "Creative Math, or Just Fuzzy Math?", the picture, entitled "Your New Math Book" whose blank pages suggested a lack of content, and the side-bar, a box filled with apparent jargon on "Answers and Solution Techniques", all created a negative impression of the goals of the reform movement. I would like to address those in this article, and to comment on the relationship of the standards approach to constructivism.

I began my presentation with a quiz, asking my audience the question which appeared in the box: "I just checked out a library book that is 1,344 pages long! The book is due in 3 weeks. How many pages will I need to read a day to finish the book in time?"

I gave them 15 seconds to come up with an answer, and then selected a dozen people at random to provide their solutions; as I anticipated, this generated a variety of solutions. Some gave the exact answer of 64 (1,344 divided by 21), but most gave answers like "about 60" or "around 70". From a show of hands, we learned that only about one-third of the audience had obtained an exact answer; the others had estimated the answer, even though they were all research mathematicians and mathematics teachers and they were all familiar with the techniques of division.

The unnamed author of the side-bar noted, perhaps nostalgically, that "The old way to solve the problem would be to use the algorithm for long division: 1,344 divided by 21." The question is: Who would actually solve the problem that way? I would venture a guess that fewer than 10% of your [i.e., the New York Times'] readers aged 40 and older have used long division even once in the past decade. Anyone who needs to solve a problem like that nowadays would use a calculator. So a fundamental issue in mathematics education is whether we should stress a technique which most students will never master and few will ever use. I will return to this question later.

How did the mathematicians in my audience get their answers? Here are three examples of the kind of thinking that took place: 1,344 divided by 21 is about 1,400 divided by 20, and that's 70; 13 divided by 2 is between 6 and 7, so the answer is a number that starts with 6; since 60 times 20 is 1200, the answer is in the 60s. After discussing the different ways we solved the problem, I showed them the next portion of the side-bar: "The MathLand curriculum guide calls for a new approach, explaining that "division in MathLand is not a separate operation to master, but rather a combination of successive approximations, multiplication, adding up and subtracting back, all held together with the student's own number sense."" My audience all identified with "division in MathLand", because that's how we actually do division.

It is important for your readers to understand that what the Mathland curriculum guide calls for is not a "new approach", but an approach which reflects the thinking of successful users of mathematics. The phrase "adding up", for example, describes what actually happens when we apply the standard method to divide 21 into 1,344:

[insert here the actual long division of 21 into 1344]

When we put 6 on top of the line, we are saying, in shorthand, that when we divide 21 into 1,344, our initial approximation to an answer is 60. Then when we divide 21 into the remainder of 84 and put a 4 on top of the line next to the 6, we are saying that in addition to the 60 times that 21 goes into 1344, it also goes in 4 times. Altogether, then, 21 goes into 1,344 a total of 60 + 4, or 64 times. Here we see "successive approximations", "multiplication", and "adding up", all used to help understand the process of long division. So what appeared to be "jargon in the box" is actually an accurate (though technical) portrayal of what should be involved in learning division.

The MathLand problem has another interesting feature which should not be overlooked. It is an example of a problem where an approximate answer, indeed an over-approximation, is preferable to an exact answer. You wouldn't expect the child to stop reading after exactly 64 pages, ignoring chapter divisions, and you wouldn't expect the child to read about 60 pages a day, lest she come up short after three weeks; the most sensible answer would be to read about 70 pages each day. Recognizing what kind of an answer is appropriate is an important part of problem solving.

Earlier, I conjectured that fewer than 10% of adults over 40 actually use long division. One reason of course is the ubiquity of calculators. But another reason is that only a few adults can explain why long division works. They may not explain it as I did in the paragraph above; they likely have their own explanation of how it works -- through a process of what is disparagingly referred to as "inventing personal methods of long division" in one of the articles. Those who can't explain it most likely can no longer do it. And, that is my understanding of "constructivism". If we have been able to construct our own understanding of a mathematical concept or our own explanation of a mathematical technique, then it is ours; and it we don't, it is gone. Now I am not a professor of mathematics education, so I don't know whether this accurately captures the notion of constructivism, but it is the notion of constructivism that I have constructed for myself.

Constructivism focuses on "understanding" and "explaining", rather than simply on memorizing or doing. Those who advocate constructivism do not insist that one shouldn't "memorize" or "do" until one can "understand" or "explain", or, beyond that, that one shouldn't "memorize" or "do" at all. (There are of course "radical constructionists" who hold those views.) But many teachers and curriculum leaders seem to think that constructivism implies that children should be exempted from learning the multiplication table. In fact, in order to solve the division problem above, you have to know and use many multiplication facts quickly. Should children learn the multiplication table? They certainly should.

Has the use of calculators affected children's learning of the multiplication table? It has if teachers haven't required them to know their multiplication facts and to solve problems regularly using those facts. Children should be using calculators, and they should be learning to multiply without them. That's common sense. When people complain about the damage caused by the use of calculators, I imagine similar complaints when the use of the abacus became widespread thousands of years ago. True, we lose some skills when technology advances; but that's a trade-off that we're willing to accept. I would guess that none of your readers has used in the last decade the method for finding square roots that I learned in school, and that no one has mourned its loss. Will that be the fate of long division?

What should be the goals of mathematics education? The point of the "standards approach" is first, to determine what those goals are, and second, to try to achieve those goals. We cannot achieve our goals until we determine what they are. What do we want all children to know and be able to do? There may be disagreement with the current answer to that question, but presumably we should all agree that an answer to that question would be of importance.

Standards are important because they describe what we value, those elements of education that we believe are critically important to the child's future. Standards are important because they can raise expectations for all students. All students can achieve more than they are now expected to achieve. Teachers and parents get, at best, what they expect. Standards can convey to both teachers and parents the goals that they should set for their children. In New Jersey, the State Board of Education last year [i.e., in 1996] adopted standards in seven content areas, and cross-content workplace readiness standards. The mathematics standards provide high achievable standards for all students, and they encourage all students to be continuously challenged and enabled to go as far mathematically as they can.

Although the mathematics standards of the National Council of Teachers of Mathematics are influenced by the constructivist approach -- they do emphasize teaching for understanding, and they do encourage children to work together to help build each other's understanding -- that is not all they are. They attempt to describe what mathematics children at various ages should know and be able to do. They focus on solving problems, not the one- or two-step problems like the one above, but problems that originate in real world situations. (Example: What size airconditioner is needed to cool the classroom?) They insist that children learn how to reason mathematically and explain their reasoning. And they have energized teachers to expect more of their students.

My involvement with K-12 education began with the observation that students entering college were not able to do what we expected of them. In the early 80s I was Director of the Undergraduate Program in Mathematics at Rutgers (New Brunswick), at which time we instituted a placement test for all incoming students. To our amazement and dismay, about 20% of those taking the test were placed into non-credit remedial courses and an additional 20% were placed into a high-school level precalculus course; this despite the fact that at that time Rutgers required three years of college preparatory high school mathematics. Changing that statistic was the initial impetus for my getting involved in education. I mention that here because, in her article, Lynn Chaney mentions parents' complaints "about high school graduates who get A's and B's in whole-math classes and have to do remedial work in college." That's nothing new.

[Interestingly, there is no such approach as "whole math". The term was apparently invented by people in the California contingent, who are trying to associate the mathematics reforms with the failed results of implementing an extreme "whole language" approach to reading and writing in California.]

And now back to the question of what goes into "Your New Math Book". As noted earlier, the multiplication facts certainly are still there, although students will also need to know what they mean. They should be able to represent multiplication facts visually; for example, the product of 5 and 7 can be pictured as an array of dots arranged in five rows and seven columns. An elementary school teacher recently told me that she never knew why the "square numbers" -- 1, 4, 9, 16, 25, 36, etc. -- were referred to in that way. That means that she had never seen the multiplication facts explained visually, since square numbers are those which correspond to square arrays.

Does long division belong in "Your New Math Book"? Here the answer is not so clear. We certainly expect that every child should understand division and be able to use it appropriately. We certainly expect that every child should be able to recognize -- mentally and quickly -- that dividing 328 into 78,965 results in an answer which is between 200 and 300. (Can you do that?) That requires understanding of the various components of division mentioned earlier, as well as doing long division using single-digit divisors. (It also requires mastery of the powers of ten.)

But should teachers spend the time required to assure that all students have mastered the traditional method of dividing 328 into 78,965 using long division? On the one hand, that will be a skill that they are unlikely to use, since they will certainly use a calculator. (Calculations is what calculators are designed to do!) And we know that a substantial percentage of students never achieve and retain mastery of the technique. Should we not rather focus our energies on teaching them to check whether they have accurately entered the data into the calculator by first making an estimate of the answer? This way, if they divide 328 into 78,965 and get an answer that is not between 200 and 300, they will know that an error has been made, and that they will have to repeat the procedure. On the other hand, the method of long division is a neat way of encapsulating a lot of thinking about division into a shorthand procedure. As such, it merits our attention and admiration. It is part of our culture. Does that mean that it belongs in the book? Yes, but it will not be treated the same way as it was a generation ago.

August 17, 1997

This article by Joseph G. Rosenstein, professor of mathematics at Rutgers University, was submitted to but not published by the New York Times.