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RECURSIVE LINEAR ORDERINGS

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RESUME.-

Dans ce bref discours sur les chaînes récursives, je voudrais vous présenter certaines questions qu'on se pose et vous donner un échantillon des méthodes et techniques qu'on emploie pour répondre à ces questions. Le contexte de ce discours est fourni par la question générale suivante:

Etant donné un théorème combinatoire concernant les chaînes, est-il vrai effectivement? Si non, comment peut-on mesurer l'effectivité du résultat ?

Un exemple de théorème combinatoire vrai effectivement est le résultat de Cantor selon lequel toute chaîne dénombrable est isomorphe à un sous-ensemble des nombres rationnels. On peut modifier la démonstration standard de ce résultat pour décrire un algorithme qui produit le théorème suivant:

Toute chaîne récursive est récursivement isomorphe à un sous-ensemble récursif des nombres rationnels.

Remarquez que tout emploi du mot "récursif" doit être précisément défini; ainsi par exemple, la phrase "chaîne récursive" entraîne l'existence d'un algorithme au moyen duquel on peut déterminer si a < b ou non, et la phrase "récursivement isomorphe" entraîne l'existence d'un algorithme au moyen duquel on peut calculer les valeurs de la fonction.

Dans l'exemple que j'ai cité ci-dessus, la démonstration traditionnelle peut être effectivisée; il y a des exemples où cela n'est pas possible. Dans ces cas-là on doit ou fabriquer un nouvel algorithme ou démontrer qu'il n'existe pas d'algorithme convenable. Par exemple, la démonstration standard que tout ensemble bien ordonné a une extension linéaire bien ordonnée ne peut pas s'effectiviser. Cependant, un autre algorithme, que j'ai décrit avec H. Kierstead, démontre que:

Tout ensemble bien ordonné qui est récursif a une extension linéaire qui est bien ordonnée et récursive.

La question de savoir si tout ensemble ordonné qui est dispersé et récursif a une extension linéaire qui est dispersée et récursive reste ouverte.

Il n'y a pas toujours une manière unique d'effectiviser une formulation combinatoire. Par exemple, une autre manière d'approcher l'exemple mentionné ci-dessus est de demander si tout ensemble ordonné qui est récursif et récursivement bien ordonné a une extension linéaire qui est récursive et récursivement bien ordonnée. ("Récursivement bien ordonné" signifie que l'ensemble ordonné n'a aucune suite infinie récursive qui soit décroissante). Cette approche mène plutôt à une solution négative, due à moi-même et à R. Statman:

Il existe un ensemble ordonné qui est récursif et récursivement bien ordonné, mais qui n'a pas d'extension linéaire qui soit récursive et récursivement bien ordonnée.

Après avoir introduit la hiérarchie arithmétique de Kleene, nous répondons à l'autre moitié de la question générale: Etant donné un ensemble ordonné qui est récursif et récursivement bien ordonné, quelle complexité l'extension linéaire désirée doit-elle posséder ?

Tout ensemble ordonné qui est récursif est récursivement ordonné a une extension linéaire qui est récursivement bien ordonnée et est située au niveau Δ_2 dans la hiérarchie arithmétique de Kleene.

La démonstration de ce théorème est un exemple de l'utilisation des "argumentations diagonales" dans la théorie de la récursivité.

Les versions effectives de quelques autres résultats combinatoires sont discutées dans l'article y compris le théorème de Dushnik et Miller selon lequel toute chaîne dénombrable peut s'abriter dans un sous-ensemble propre d'elle-même.

Ces sujets sont discutés à fond dans mon livre Linear Orderings (Academic Press, 1982) dans le chapitre intitulé, comme on peut s'y attendre, Linear Orderings and Recursion Theory.

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The framework for this survey article about recursive linear orderings is provided by the following general questions: Given a combinatorial theorem about linear orderings, is it true effectively ? If not, how can the effectiveness of the result be measured ? Among the specific combinatorial theorems considered in detail are (a) Every countable linear ordering is order-isomorphic to a subset of the rationals (Cantor), (b) Every well-founded (resp., scattered) partial ordering has a well-founded (resp;, scattered) linear extension, and (c) Every countable linear ordering can be embedded into a proper subset of itself (Dushnik and Miller).

*

In this brief talk about recursive linear orderings, I would like to convey to you some sense of the questions that are raised and some of the flavor of the methods and techniques that are used in answering these questions. I will not try to be comprehensive.

The framework for this talk is provided by the following general question:

Given a combinatorial theorem about linear orderings, is it true effectively? If not, how can the effectiveness of the result be measured?

I use the phrase "linear orderings" the way others use the phrases "total orderings" or "chains." Information about linear orderings can be found in my recent book [5] on the subject; the chapter on recursive linear orderings also contains the appropriate material on recursion theory.

All linear orderings that I discuss will be countable, and therefore we need only look at subsets of Q (the rational numbers.) This is correct by the following theorem, due to Cantor:

Every countable linear ordering is (order) isomorphic to a subset of Q.

Is this combinatorial theorem true effectively?

Before answering this question, we must first formulate and explain the effective version of Cantor's theorem. Loosely speaking, the way to arrive at an effective version of a combinatorial result is to add the word "recursive" at all appropriate locations. Using this heuristic, we arrive at the following statement:

Each of the three usages of the term "recursive" requires an explanation. Let us think of a countable linear ordering as a structure $\langle N,R\rangle$ where N is the set of natural numbers and R is a binary relation on N which defines a linear ordering of N. Then we define the linear ordering $\langle N,R\rangle$ to be <u>recursive</u> if there is an algorithm which, given a and b, will determine whether aRb or bRa is correct. Similarly, a subset $A \subseteq Q$ is <u>recursive</u> if there is an algorithm which, given $a \in Q$, will determine whether or not $a \in A$. Also, a function $f:N \to Q$ is <u>recursive</u> if there is an algorithm which, given $a \in N$, will produce $f(a) \in Q$; thus to say that $\langle N,R\rangle$ is recursively isomorphic to a subset $A \subseteq Q$ means that there is a recursive function $f:N \to A$ which defines an order isomorphism between $\langle N,R\rangle$ and $\langle A,<\rangle$.

With these definitions, it turns out that the effective version of Cantor's theorem stated above is correct, and moreover that the traditional proof can easily be effectivized, since it already has an algorithmic flavor. Thus consider the induction step for $n = 6$. We assume as the induction hypothesis that

$$\text{if} \quad 3\ R\ 2\ R\ 5\ R\ 1\ R\ 4$$

$$\text{then} \quad f(3) < f(2) < f(5) < f(1) < f(4).$$

If now $3\ R\ 6\ R\ 2$, then we would choose $f(6)$ between $f(3)$ and $f(2)$, as in the diagram below:

In order for f to be defined recursively, we must specify $f(6)$ in a way which can be systematically applied to define the other values of f. The natural solution to this problem seems to be to define $f(6)$ to be the midpoint of the interval, but this leads to some difficulty for the following reason. We have specified that the image of f be a recursive subset of Q so that we must be able to provide an answer to questions like "Is $(f(5) + f(1))/2$ in $f[N]$?" Now if for some $a \in N$ we have $5\ R\ a\ R\ 1$, then indeed the midpoint of the interval $[f(5), f(1)]$ will be in Q, but otherwise it will not be. But the question of whether there is such an element $a \in N$ cannot be answered recursively, for the recursiveness of R only enables us to decide simple issues like the relative position of two elements, but not more complicated questions like the non-emptiness of an interval. Thus another definition of $f(6)$ must be produced.

Let us specify in advance a fixed canonical enumeration $\{q_n | n \in N\}$ of Q (which will be referred to elsewhere in this paper) and arrange the construction so that if q_n is none of $f(1), f(2), \ldots, f(n)$, then q_n is not in $f[N]$ at all; then to determine whether or not $b \in f[N]$ we need only determine b's place in the enumeration and, assuming that $b = q_m$, carry out the first m applications of the algorithm, and at that time determine whether or not $b \in f[N]$. To achieve this effect, we define $f(6)$ to be the first rational number on the list $q_6, q_7, q_8, \ldots$ which is between $f(3)$ and $f(2)$, and similarly for each other value of n. This algorithm then has all the required properties.

In the example above the traditional proof can be effectivized; there are examples where it cannot. In such cases, one must either produce a new algorithm or prove that no suitable algorithm exists.

As a second example of a combinatorial result, I will, in deference to the organizers of this conference, discuss extendability of partial orderings. (I am reporting here on discussions with H. Kierstead and R. Statman.) Consider the following statement:

> *If A is a well-founded partial ordering, then A has a well-founded linear extension.*

Is this true recursively? That is, is the following statement correct?

> *If A is a well-founded recursive partial ordering, then A has a well-founded recursive linear extension.*

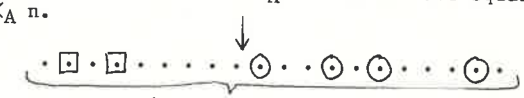
Here of course we are thinking of a recursive partial ordering as a relation R on N for which there is a suitable algorithm.

The standard proof of this result (see for example Bonnet and Pouzet [1]) proceeds by "levels." That is, we define A_0 to consist of all minimal elements of A, and proceed inductively to define A_α to consist of all minimal elements of $A - \bigcup\{A_\beta \mid \beta<\alpha\}$. We now observe that the elements of each A_α are pairwise incomparable and that this procedure terminates by some ordinal $\gamma<|A|^+$ so that $A = \bigcup\{A_\alpha \mid \alpha<\gamma\}$. We then well-order each A_α to get B_α and we define $B = \Sigma\{B_\alpha \mid \alpha<\gamma\}$.

This proof clearly cannot be effectivized. In fact, we cannot even begin this way since there is no way of telling recursively which elements of A are minimal.

There is however another algorithm which works. This algorithm and the proof that it works are due to myself and H. Kierstead.

At stage n, we assume that we have already defined whether $a <_B b$ holds, for all $a, b < n$, and that $<_B$ is a linear ordering of $\{1, 2, \ldots, n-1\}$ which extends $<_A$ on that set. Our task at stage n is to place n properly among $\{1, 2, \ldots, n-1\}_B$. In the picture below, we have circled those elements m of $\{1, 2, \ldots, n-1\}$ for which $n <_A m$ and we have squared those elements m for which $m <_A n$.



$\{1, 2, \ldots, n-1\}_B$

We could place n anywhere between the largest squared element and the smallest circled element. We choose to place n at the location indicated by the arrow; that is, n is placed just below the B-smallest $m < n$ for which $n <_A m$. Thus n is placed as high as possible among $\{1, 2, \ldots, n-1\}_B$.

It is clear that with this algorithm we do end up with a recursive linear extension of A -- for to determine whether $a <_B b$ or $b <_B a$, we need only reproduce the first k applications of this algorithm where k is the larger of a and b.

Note that although each element is placed as high as possible at the appropriate stage, it is possible that at stage a, a is placed below b with $b < a$, even though $a \not<_A b$. In the diagram above such b are uncircled elements which are to the right of circled elements. That is, if $a|_A b$, $a > b$, and $a <_B b$, then at stage a we placed a below b because $a <_A c$ for some c for which $c < a$ and $c <_B b$. A sharper observation is that c can be chosen so that, in addition, $c < b$; this fact can be proved by induction on a.

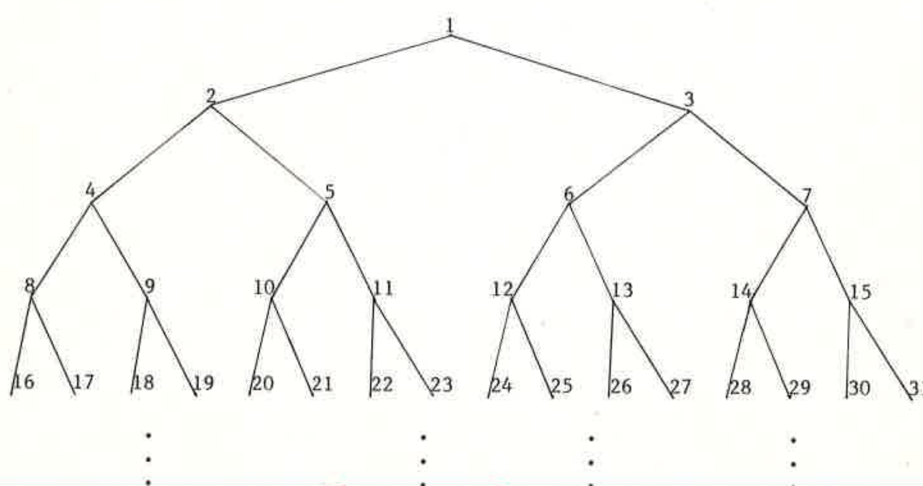
Now suppose that B is not well-founded, so that there is a subset $\{b_1, b_2, b_3, \ldots\}$ of B so that $b_1 >_B b_2 >_B b_3 >_B \cdots$. We may assume without loss of generality that $b_1 < b_2 < b_3 < \cdots$. For each $i < j$, either $b_i >_A b_j$ or $b_i |_A b_j$, so by Ramsey's theorem and the assumption that A is well-founded, we may also assume that $b_i |_A b_j$ for every i and j. For each $i > 1$ we thus have $b_i |_A b_1$, $b_i > b_1$, and $b_i <_B b_1$ so that for each $i > 1$ there is some c_i such that $c_i < b_1$ and $b_i <_A c_i <_B b_1$. Since each $c_i < b_1$, there must be some $c_1 < b_1$ such that $b_i <_A c_1 <_B b_1$ for infinitely many b_i; we may of course assume that c_1 is chosen minimally with this property. By eliminating all other b_i, we may thus assume that the original sequence satisfies $b_i <_A c_1 <_B b_1$ for all $i > 1$.

If we continue this procedure inductively, we conclude that for the original sequence $b_1 >_B b_2 >_B b_3 >_B \ldots$ there is a sequence $c_1, c_2, c_3, \ldots$ such that for all $i > j$ we have $b_i <_A c_j <_B b_j$ and that each c_j is chosen minimally (so that there is no $d < c_j$ such that $b_i <_A d$ for infinitely many b_i with $i > j$). Because of the minimality, and since $c_i <_B b_i <_A c_j$ for $i > j$ implies that $c_i \neq c_j$, we know that $c_1 < c_2 < c_3 < \cdots$.

We now claim that $c_1 >_A c_2 >_A c_3 >_A \ldots$, contradicting the assumption that A is well-founded. Indeed, since $c_{j+1} <_B b_{j+1} <_A c_j$, we know that $c_{j+1} <_B c_j$. If $c_{j+1} \not<_A c_j$ then we must have $c_{j+1} |_A c_j$. Since $c_{j+1} > c_j$, it follows that for some $d < c_j$, we get $c_{j+1} <_A d <_B c_j$. But then $b_i <_A c_{j+1} <_A d <_B c_j$ for all $i > j+1$. This contradicts the choice of c_j since $d < c_j$. Hence $c_{j+1} <_A c_j$ for all j, contradicting the assumption that A is well-founded. Hence B is well-founded, completing the proof.

As noted by several people, this algorithm provides a new proof that if κ is an infinite cardinal number and $\kappa \not\leq A$ than there is a linear extension B of A for which $\kappa \not\leq B$.

It is interesting to see what happens if this particular algorithm is applied to a partial ordering which is not well-founded. Let A be the full binary tree, labelled as in the diagram:



It is easy to see that if this tree were extended to a linear ordering using the algorithm presented above, then the even numbered points would form a set of order type η (of the rationals). Indeed for each n, the 2^{n-1} even numbers at level n of the tree would, in this linear extension, be located in the 2^{n-1} intervals formed by the $2^{n-1}-1$ even numbers at higher levels of the tree. Of course, one can extend this partial ordering to a scattered linear ordering by using the opposite algorithm, namely, by locating each point at the *lowest* position possible.

The example above arises when one asks whether every scattered recursive partial ordering has a scattered recursive linear extension. The combinatorial version of this question has a positive answer, due to Bonnet and Pouzet (see [1]), and also to Galvin and MacKenzie. Whether the effective version is true is an open question.

Open question: Does every scattered recursive partial ordering have a scattered recursive linear extension?

The astute reader will perhaps have noticed that in giving the effective version of our second example we did not quite follow the prescription laid down in our first example, namely, to add the word "recursive" at all appropriate locations. In the theorem we assumed that A had *no* subset of order type ω^* and showed that then B had *no* subset of order type ω^*. What if we assume only that A has no *recursive* subset of order type ω^*; can we then extend A to a recursive linear ordering which has no *recursive* subset of order type ω^*?

Let us then say that A is *recursively well-founded* if A has no recursive subset of order type ω^*. The conclusion below, due to myself and R. Statman, answers the question above:

There is a recursive partial ordering which is recursively well-founded but has no recursively well-founded recursive linear extension.

The proof of this result--by constructing a suitable counterexample--has two steps. The first step involves constructing a recursive binary tree--a sub-ordering of the tree above--which though infinite (and therefore contains an infinite path by König's Theorem) contains no infinite recursive path. This kind of construction is familiar to recursion theorists; I will say more about how one constructs such a tree later in this paper.

The second part involves showing that such a partial ordering cannot be extended to a recursively well-founded recursive linear ordering. Assume then that A is a recursive subtree of the full binary tree which is recursively well-founded and assume that B is a recursive linear extension of A. We will show that B has a recursive subset of order type ω^* by constructing a recursive ω^*-sequence of elements of B. This will of course prove the result.

The method of constructing such a sequence might be called a B-first search through A, generalizing the notions of breadth-first search (i.e. level search) and depth-first search (i.e. greedy search) through A which correspond to particular linear extensions B of A. At the end of stage n we will have defined a descending B-sequence $\{a_1, a_2, \ldots, a_n\}$ and a pool P_n of elements of A from which a_{n+1} will be selected. We assume as part of the induction hypothesis that $\{a_1, a_2, \ldots, a_n\}$ forms a terminal part of the tree A, that is, for each i, if b is above a_i on the tree, then $b = a_j$ for some $j < i$, and that the pool P_n consists of all immediate A-predecessors of elements of $\{a_1, a_2, \ldots, a_n\}$ except those which are already in and let P_1 be the A-predecessor of a_1. Proceeding inductively we let a_{n+1} be the B-largest element of P_n and we obtain P_{n+1} from P_n by deleting a_{n+1} and adding its A-predecessors. This completes the construction of what is clearly a recursive subset of A. To show that this set has order type ω^* in B, it suffices to show that $a_{t+1} <_B a_t$ for each t. But otherwise $a_{t+1} >_B a_t$, for since $a_t \notin P_t$ we cannot have $a_{t+1} = a_t$. Also $a_{t+1} \notin P_{t-1}$ for otherwise we would have chosen it to be a_t. But $a_{t+1} \in P_t$ so it must be an A-predecessor of a_t, so that $a_{t+1} <_A a_t$. This contradicts the assumption that B extends A. Hence $a_{t+1} <_B a_t$ for all t and B is not recursively well-founded.

We have thus presented two effective versions of the combinatorial fact that every well-founded partial ordering has a well-founded linear extension. One is correct and the other fails. I mentioned earlier that I do not know whether the η version of the first result is correct; I also do not know whether the η version of the second result is correct.

Open question: Does every recursively scattered recursive partial ordering have a recursively scattered recursive linear extension?

Now that we have a combinatorial statement about linear orderings that is not true effectively, we go back and ask how the effectiveness of the original result can be measured.

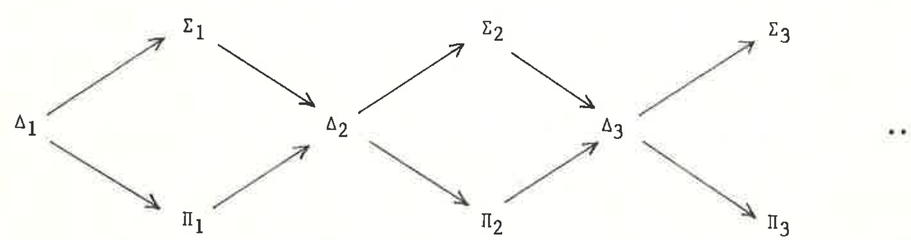
To discuss this question, we have to introduce the classification of the arithmetical sets. We say that a set K of numbers is a Σ_n-set if there is a recursive predicate $R(x_1, x_2, \ldots, x_n, y)$ such that

$$m \in K \longleftrightarrow (\exists x_1)(\forall x_2)(\exists x_3)\cdots(Q_n x_n)\, R(x_1, x_2, \ldots, x_n, m)$$

where the n quantifiers are alternately $\exists$ and $\forall$, starting with $\exists$. Similarly, we say that K is a Π_n-set if there is a recursive predicate $R(x_1, x_2 \ldots, x_n, m)$ such that

$$m \in K \longleftrightarrow (\forall x_1)(\exists x_2)(\forall x_3)\cdots(Q_n x_n)\, R(x_1, x_2, \ldots, x_n, m)$$

where again the quantifiers alternate, this time starting with $\forall$. A set K is a Δ_n-set if it is both a Σ_n-set and a Π_n-set. A set is called an arithmetical set if it is a Σ_n-set or a Π_n-set for some n. That these sets form a hierarchy was shown by S.C. Kleene (for reference see [5]); that is, in the diagram below all inclusions (indicated by arrows) are proper inclusions.



For example, a set E is recursively enumerable if there is a Turing machine which will produce all of its members; that is, if for some t,

$$m \in E \longleftrightarrow (\exists s)\ T(t, s, m)$$

where $T(t, s, m)$ is the recursive predicate which says that "Turing machine number t will produce m within s steps." Now for each fixed t, the predicate $T_t(s, m) \equiv T(t, s, m)$ is a recursive predicate $R(s, m)$, so that $m \in E \longleftrightarrow (\exists s)\ R(s, m)$. Thus every recursively enumerable set is a Σ_1-set; the converse is also correct. A set is recursive if and only if both it and its complement are recursively enumerable, or Σ_1-sets. Note that a set is a Σ_n-set if and only if its complement is a Π_n-set, so that the Δ_1-sets are precisely the recursive sets.

We let W_t denote the t'th recursively enumerable set: $W_t = \{m \mid (\exists s) T(t, s, m)\}$. (Hidden in the notation is the Enumeration Theorem which says that the recursively enumerable sets can themselves be recursively enumerated.) For which t is W_t non-empty? This set is Σ_1 since

$$W_t \text{ non-empty} \longleftrightarrow (\exists m)(\exists s)\ T(t, s, m).$$

(In this predicate, the two existential quantifiers can be combined into one by using the recursive function $f(n, x) = (x)_n$ = the power of the n'th prime in x. Thus the above is equivalent to $(\exists x)\ T(t, (x)_1, (x)_0)$ which can be simplified to $(\exists x)\ R(t, x)$.) For which t is W_t infinite? This set is Π_2 since

$$W_t \text{ infinite} \longleftrightarrow (\forall n)(\exists m)\ (m > n \wedge m \in W_t)$$

which can be rewritten in the appropriate Π_2 form.

One additional topic that we need to discuss briefly is the use of an oracle. During a construction we may find it necessary to refer to a set A which is external to the construction, and inquire whether given numbers are in A. In that case the set or function constructed is said to be <u>recursive in A</u>. It turns out that if A is a Σ_n-set or a Π_n-set, then sets recursive in A are all Δ_{n+1}-sets. Thus, for example, if during a construction we need to know for each t whether W_t is infinite, then whatever we construct will be a Δ_3-set; in this case, the oracle being consulted is $\{t \mid W_t \text{ is infinite}\}$.

The arithmetical hierarchy provides a useful measure of the complexity of given sets or constructions. The value judgment it propounds is that the lower in the hierarchy the outcome of a particular construction, the better the construction.

We now return to our previous question. Given a recursively well-founded recursive partial ordering A, how complicated need a desirable linear extension B of A be?

In order for B to be recursively well-founded, it is sufficient to guarantee that no W_t provides a B-decreasing sequence of elements of A. How do we do this? Suppose that after s steps we have enumerated (in order) the elements $a_1, a_2, \ldots, a_k$ of W_t. If $a_j <_B a_i$ for some $j < i$ then W_t will not be a B-decreasing sequence of elements of A. Otherwise, if we sit idly by, it is possible that we will eventually have $a_1 >_B a_2 >_B \ldots >_B a_k$ and that, even worse, this pattern will continue until the very end. So we must find some a_i and a_j with $i < j$ (either from those elements so far generated or from those not yet generated) for which a_i is not yet B-larger than a_j, and define $a_i <_B a_j$. Since at each step of the construction of B we make only finitely many decisions about A-incomparables, if W_t is infinite we will always be able to find such an i and j, for otherwise we would be able to find an ω^*-sequence of elements of A.

The trouble is that we have to do the same thing for each W_t and so we cannot wait around to find out if W_t is infinite. If it is, we will eventually find a suitable i and j, but if it is not, then our search will go on forever and no B will be constructed at all. If at a given stage of the construction we want to dispose of W_t then we will need to know whether or not W_t is infinite; if it is, we will search for the right a_i and a_j, and if it is not, we will go on to W_{t+1}. Thus the construction of B will be recursive in a Π_2 set. The reader may have observed that we can use a simpler oracle, namely, one that answers whether W_t has another element. Since this question is Σ_1, the construction of B will be recursive in an Σ_1-set. We thus come to the following conclusion:

If A is a recursively well-founded recursive partial ordering, then A has a recursively well-founded Δ_2 linear extension.

By a similar argument it is possible to show that

If A is a recursively scattered recursive partial ordering, then A has a recursively scattered Δ_2 linear extension.

The proof we presented above is an example of the use of "diagonal arguments" in recursion theory.

Before going on to the next example, let me pause to point out how to construct an infinite recursive binary tree which is recursively well-founded; such a tree was used in an earlier argument. Start generating the full binary tree. At a certain time, we may decide to close off a certain node; by this we mean that nothing new will go below that node, so that below that node the tree will remain finite. When do we make such a decision? We simultaneously begin enumerating all W_t, and if it happens at some point that we have enumerated t elements of a particular W_t and those elements form a descending chain so far on the tree, then we close off the lowest of the t nodes. The reader should verify that the tree that results has the desired properties.

We now consider a third example of a combinatorial theorem about linear ordering, this one due to Dushnik and Miller [2].

Any countable linear ordering can be embedded into a proper subset of itself.

Can any recursive linear ordering be recursively embedded into a proper recursive subset of itself?

Those of you who have seen the proof of Dushnik and Miller's theorem will recall that in the scattered case the desired embedding was the identity outside of a subset of order type ω (or ω^*) and was the successor (or predecessor) map on that subset. The twist came in the non-scattered case where a subset of order type η was selected and A was embedded into that subset using Cantor's theorem. More precisely, this procedure was needed only when A is a dense sum of finite linear orderings.

Thus we need to find a dense subset of a dense sum of finite linear orderings. Can we make such a selection recursively? For example, suppose that A is a recursive linear ordering of order type $2\cdot\eta$. The right-hand endpoints form a subset of order type η but there is no way to tell recursively which points are right-hand endpoints. (Show that this is a Δ_2-set.) Nevertheless, the effective version of Dushnik and Miller's theorem is correct in this

Any recursive subset A of Q of order type $2\cdot\eta$ can be recursively embedded into a proper recursive subset of itself.

To prove this we enumerate $A = \{a_1, a_2, a_3, \ldots\}$ in order of discovery and begin the definition of a function $P: A \dashrightarrow A$ for which $a_1 \notin P[A]$ by setting $P(a_1) = a_2$. How do we define $P(a_2)$? If for example $a_2 > a_1$ we can wait for some $a_m > a_2$ and then define $P(a_2) = a_m$. But that will give us problems later if a_m is the successor of a_2 but a_2 is not the successor of a_1. Instead we wait for an a_m which is not the successor of a_2. How to do this systematically we leave for the reader. (A more complete discussion of the proofs of this and subsequent results in this paper can be found in my book [5].)

The same argument works whenever A is a dense sum of bounded finite linear orderings. The reader might try his hand at the apparently open question of what happens when A is an arbitrary recursive dense sum of finite linear orderings.

We started this discussion by mentioning how easily Dushnik and Miller's theorem works in the scattered case. This is far from true recursively, since one cannot identify successors or determine the size of intervals recursively even if A has order type ω. The following result, due to L. Hay and Rosenstein, is proved in [5].

There is a recursive subset A of Q of order type ω for which there is no recursive map from A to A other than the identity.

There are several interesting questions which were raised in the above discussion which merit further attention. Does every non-scattered recursive subset of Q have a recursive subset of order type η? The embedding of $2\cdot\eta$ into itself does yield a recursive subset of order type η but that is the best that can be done. That is, it is possible to construct a recursive subset of Q of order type $\xi\cdot\eta$ (ξ is the order type of the integers) which has no Δ_2 dense subset (Lerman and Rosenstein [4]); the best one can thus say is that such a set has a Π_2 dense subset (try to prove this). I suspect, but I cannot prove, that there is a recursive non-scattered linear ordering which has no arithmetical dense subset.

Open question: Is there a recursive non-scattered linear ordering which has no arithmetical dense subset?

A possible line of attack would be to construct for each n a recursive non-scattered linear ordering of order type $\xi^n\cdot\eta$ which has no Δ_{2n} recursive dense subset, and then perhaps one of order type $\xi^\omega\cdot\eta$ which has no arithmetical dense subset.

Since there is no end to the combinatorial results about linear orderings (and other structures), there is also no end to the questions of effectiveness of combinatorial results that can arise. I will discuss briefly two more types of questions. The first is the following: Given a recursive linear ordering which has an automorphism, must it have a recursive automorphism? In my book I construct examples of recursive linear orderings of order type ξ and $2\cdot\eta$ which have no recursive automorphisms. (The latter construction uses a priority argument, which is a more sophisticated type of diagonal argument.) A recent thesis by Steven Schwarz [6] contains the following definitive result: Let α be a recursive order type (i.e. the order type of a recursive linear ordering). Then there is a recursive subset A of Q of order type α which has no

Finally, I should discuss the oldest result concerning effective versions of theorems about linear orderings. Consider the classical fact that every infinite linear ordering has either a subset of order type ω or a subset of order type ω^*. What is the combinatorial version of this fact? Tennenbaum showed a number of years ago that there is a recursive subset of Q of order type $\omega + \omega^*$ which has no recursive subset of order type ω or ω^*; such a set can be thought of as effectively finite. Watnick [7] extended Tennenbaum's construction to show that for every recursive order type α there is a recursive subset of Q of order type $\omega + \xi \cdot \alpha + \omega^*$ which has no recursive subset of order type ω or ω^*. Looking in a different direction, it is possible to show that every recursive linear ordering has a recursive subset of order type ω, ω^*, $\omega + \omega^*$, or $\omega + \xi \cdot \eta + \omega^*$; Lerman [3] showed that this was best possible by constructing a recursive linear ordering of the last order type which has no recursive subset of any of the other three order types.

As I have mentioned several times already, these results and constructions are discussed at some length in my book [5], in the chapter titled, not surprisingly, Linear Orderings and Recursion Theory.

I hope that in this talk I have succeeded in conveying some of the flavor of the subject of recursive linear orderings.

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