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Theories which are not $\aleph_0$-Categorical

J. G. Rosenstein

A well-known theorem (Engeler, Ryll-Nardzewski, Svenonius) characterizes the $\aleph_0$-categorical theories T as those for which $B_n(T)$ is finite for each n (where $B_n(T)$ is the Boolean algebra of equivalence classes of well-formed formulas in the variables $x_1,\ldots,x_n$ with respect to the equivalence relation

$$T \vdash (x_1)\ldots(x_n)(\varphi(x_1,\ldots,x_n) \Longleftrightarrow \psi(x_1,\ldots,x_n)) .$$

One immediately asks whether a better result is possible--namely, whether there is an m such that T is $\aleph_0$-categorical whenever $B_m(T)$ is finite. To deny this, it is sufficient to exhibit for each n a system M_n for which $B_n(T(M_n))$ is finite but $B_{n+1}(T(M_n))$ is infinite. That is precisely the purpose of this paper.

The system M_n is a partial ordering in which transitivity is vacuously satisfied--i.e., no element satisfies both $(Ey)(x<y)$ and $(Ey)(y<x)$, so that M_n consists of a top set and a bottom set with each element on the bottom being less than certain elements on the top. Thus the universe of M_n can be written as the disjoint union $\{a_i\}_{i\in N} \cup \{b_j\}_{j\in N}$, and if $d<c$ then d is b_j for some (unique) j and c is a_i for some (unique) i. We shall denote $\{b_j \mid b_j < a_i\}$ by $\underline{a}_i$.

To guarantee that $B_{n+1}(T(M_n))$ is infinite, we construct M_n so that for each k the well-formed formula $(E_k z)(z<x_1 \wedge \ldots \wedge z<x_{n+1})$ is satisfied in M_n. ($(E_k z)(\ldots)$ is of course an abbreviation for a well-formed formula which would normally be interpreted "there are exactly k distinct z's such that...")

The same shouldn't happen to $B_n(T(M_n))$ so we will be particularly careful and have $\underline{a}_{i_1} \cap \underline{a}_{i_2} \cap \ldots \cap \underline{a}_{i_n}$ infinite for any choice of $i_1, i_2, \ldots, i_n$. Of course more than that will be necessary in order to make sure that $B_n(T(M_n))$ is finite; we shall in fact arrange matters so that whenever two n-tuples $\langle r_1,r_2,\ldots,r_n\rangle$ and $\langle s_1,s_2,\ldots,s_n\rangle$ have the same "configuration" (i.e., for each i and j, $r_i = r_j$ if and only if $s_i = s_j$, $r_i < r_j$ if and only if $s_i < s_j$, and r_i is a top (bottom) element if and only if s_i is a top (bottom) element, then there is an automorphism π of M_n such that $\pi(r_i) = s_i$ for $1 \leq i \leq n$. This symmetry guarantees that the well-formed formula which describes the "configuration" of the n-tuple is an atom of $B_n(T(M_n))$, and, since there are but a finite number of "configurations" of n elements, the number of such atoms is finite; but since any n-tuple has a configuration, it satisfies some atom, so we may conclude that $B_n(T(M_n))$ is generated by these atoms, hence is finite. This symmetry will be realized by guaranteeing that for any set of top elements and any coherent way of choosing for each n-element subset $a_{i_1}, \ldots, a_{i_n}$ of these a finite subset B of $\underline{a}_{i_1} \cap \ldots \cap \underline{a}_{i_n}$ there are top elements a which simultaneously realize all of these intersections in the sense that

$$\underline{a} \cap (\underline{a}_{i_1} \cap \ldots \cap \underline{a}_{i_n}) = B$$

for each $i_1, \ldots, i_n$ and corresponding B.

Thus we shall define a relation $<$ on $\{a_i\}_{i \in N} \cup \{b_j\}_{j \in N}$ so that

(i) if $d < c$ then d is b_j for a unique j and c is a_i for a unique i,

(ii) $\underline{a}_{i_1} \cap \underline{a}_{i_2} \cap \ldots \cap \underline{a}_{i_n}$ is infinite for each $i_1, i_2, \ldots, i_n$,

(iii) $a_{i_1} \cap \underline{a}_{i_2} \cap \ldots \cap \underline{a}_{i_n} \cap \underline{a}_{i_{n+1}}$ is finite if $i_1, i_2, \ldots, i_{n+1}$ are all different,

(iv) if B is a finite subset of $\{b_j\}_{j \in N}$ then there are infinitely many i's for which $B \subseteq \underline{a}_i$,

(v) if K is finite and for each n-element subset L of K we are given a finite set $B_L \subseteq (\bigcap_{i \in L} \underline{a}_i)$ such that

$$(\bigcup_{L \in \mathscr{L}} B_L) \cap (\bigcap_{i \in \mathscr{L}} \underline{a}_i) = B_L$$

for each $L \in \mathscr{L}$ (where $\mathscr{L}$ is the set of all n-element subsets of K), then there are infinitely many distinct i's for which $\underline{a}_i \cap (\bigcap_{i \in L} \underline{a}_i) = B_L$ for each $L \in \mathscr{L}$.

Before defining the relation $<$ we shall see why the above properties do guarantee that M_n serves our purposes.

<u>Theorem</u>: Let $\langle r_1, r_2, \ldots, r_n \rangle$ and $\langle s_1, s_2, \ldots, s_n \rangle$ be two n-tuples of elements of M_n such that for each i and j, $r_i = r_j$ if and only if $s_i = s_j$, $r_i < r_j$ if and only if $s_i < s_j$ and r_i is a top (bottom) element of M_n if and only if s_i is a top (bottom) element of M_n. Then there is an automorphism π of M_n such that $\pi(r_i) = s_i$ for $1 \leq i \leq n$.

Proof: Let $\langle a_{i_1},\dots,a_{i_t}\rangle$ be an arrangement of the distinct top elements among $\{r_1,\dots,r_n\}$ and let $\langle a_{j_1},\dots,a_{j_t}\rangle$ be the corresponding arrangement of the distinct top elements among $\{s_1,\dots,s_n\}$. Let $\langle b_{i_1},\dots,b_{i_c}\rangle$ be an arrangement of the distinct bottom elements among $\{r_1,\dots,r_n\}$ and let $\langle b_{j_1},\dots,b_{j_c}\rangle$ be the corresponding arrangement of the distinct bottom elements among $\{s_1,\dots,s_n\}$.

So that we can apply (v), we select $a_{i_{t+1}},\dots,a_{i_n}$ and $a_{j_{t+1}},\dots,a_{j_n}$. Using (iv) let $a_{i_{t+1}},\dots,a_{i_n}$ be $n-t$ distinct a's for which $\{b_{i_1},\dots,b_{i_c}\}\subseteq \underline{a}$ and let $a_{j_{t+1}},\dots,a_{j_n}$ be n-t distinct a's for which $\{b_{j_1},\dots,b_{j_c}\}\subseteq \underline{a}$. We shall construct an automorphism π of M_n such that $\pi(a_{i_x})=a_{j_x}$ for $1\le x\le n$ and $\pi(b_{i_x})=b_{j_x}$ for $1\le x\le c$.

The construction of π will be done in stages. We assume that at the end of stage k we have defined two sequences $\langle a_{i_1},a_{i_2},\dots,a_{i_{n+k}}\rangle$ and $\langle a_{j_1},a_{j_2},\dots,a_{j_{n+k}}\rangle$. Let $R_k=\bigcup_{K\in\mathcal{K}}(\bigcap_{x\in K}\underline{a}_{i_x})\cup\{b_{i_1},\dots,b_{i_c}\}$ and $S_k=\bigcup_{K\in\mathcal{K}}(\bigcap_{x\in K}\underline{a}_{j_x})\cup\{b_{j_1},\dots,b_{j_c}\}$ where $\mathcal{K}$ is the set of all n+1-element subsets of $\{1,\dots,n+k\}$. (Note that R_k and S_k are finite.) We assume also that we have defined $\pi_k\colon R_k\cup\{a_{i_1},\dots,a_{i_{n+k}}\}\to S_k\cup\{a_{j_1},\dots,a_{j_{n+k}}\}$ so that π_k is 1-1 onto, $\pi_k(a_{i_x})=a_{j_x}$ for $1\le x\le n+k$, $\pi_k(b_{i_x})=b_{j_x}$ for $1\le x\le c$, and that if $b\in R_k$ then $b<a_{i_x}$ if and only if $\pi_k(b)<a_{j_x}$.

At stage $k+1$ we will define $a_{i_{n+k+1}}$, $a_{j_{n+k+1}}$, R_{k+1}, S_{k+1} and we will extend π_k to $\pi_{k+1}\colon R_{k+1}\cup\{a_{i_1},\dots,a_{i_{n+k+1}}\}\to S_{k+1}\cup\{a_{j_1},\dots,a_{j_{n+k+1}}\}$. It will be clear from our definitions that $\bigcup_k R_k=\bigcup_k S_k=\{b_j\}_{j\in N}$, that every a_i is some a_{i_x} and is some a_{j_y}, and that $\bigcup_k \pi_k$ is indeed an automorphism of M_n with the required properties.

Stage $k+1$: Assume that k is even and let $a_{i_{n+k+1}}$ be the first a_i not occurring among $a_{i_1},\dots,a_{i_{n+k}}$. (If k is odd, we let $a_{j_{n+k+1}}$ be the first a_i not occurring among $a_{j_1},\dots,a_{j_{n+k}}$ and proceed analogously to find $a_{i_{n+k+1}}$.)

Let $A_L=\underline{a}_{i_{n+k+1}}\cap(\bigcap_{x\in L}\underline{a}_{i_x})$ for each $L\in\mathcal{L}$ (where $\mathcal{L}$ is the set of all n-element subsets of $\{1,\dots,n+k\}$) and let $A_L'=A_L-R_k$. Note that the A_L' are pairwise disjoint and that the A_L are finite.

Let B_L' be a subset of $(\bigcap_{x\in L}\underline{a}_{j_x}-S_k)$ with exactly $|A_L'|$ elements (this is possible because of (ii)) and let $B_L=B_L'\cup\pi_k(A_L\cap R_k)$. Then $B_L\subseteq\bigcap_{x\in L}\underline{a}_{j_x}$ and

$$(\bigcup_{L\in\mathcal{L}} B_L) \cap \bigcap_{x\in L_o} \underline{a}_{j_x} = \bigcup_{L\in\mathcal{L}} (B_L \cap \bigcap_{x\in L_o} \underline{a}_{j_x}) = \bigcup_{L\in\mathcal{L}} ([B'_L \cup \pi_k(A_L \cap R_k)] \cap \bigcap_{x\in L_o} \underline{a}_{j_x})$$

$$= \bigcup_{L\in\mathcal{L}} (B'_L \cap \bigcap_{x\in L_o} \underline{a}_{j_x}) \cup \bigcup_{L\in\mathcal{L}} (\pi_k(A_L \cap R_k) \cap \bigcap_{x\in L_o} \underline{a}_{j_x}) = B'_{L_o} \cup \pi_k(A_{L_o} \cap R_k) = B_{L_o}$$

for each $L_o \in \mathcal{L}$. Hence by (v) we can find an $a_{j_{n+k+1}}$ distinct from $a_{j_1},\ldots,a_{j_{n+k}}$ so that

$$\underline{a}_{j_{n+k+1}} \cap (\bigcap_{c\in L} \underline{a}_{j_c}) = B_L$$

for each $L \in \mathcal{L}$. Let

$$R_{k+1} = \bigcup_{K\in\mathcal{K}} (\bigcap_{c\in K} \underline{a}_{i_c}) \cup \{r_{i_1},\ldots,r_{i_t}\}$$

$$S_{k+1} = \bigcup_{K\in\mathcal{K}} (\bigcap_{c\in K} \underline{a}_{j_c}) \cup \{s_{i_1},\ldots,s_{i_t}\}$$

where $\mathcal{K}$ is now the set of all n+1-element subsets of $\{1,\ldots,n+k+1\}$ and define

$$\pi_{k+1}\colon R_{k+1} \cup \{a_{i_1},\ldots,a_{i_{n+k+1}}\} \to S_{k+1} \cup \{a_{j_1},\ldots,a_{j_{n+k+1}}\}$$

by $\pi_{k+1} | D\pi_k = \pi_k$, $\pi_{k+1}(a_{i_{n+k+1}}) = a_{j_{n+k+1}}$, and $\pi_{k+1}\colon A'_L \to B'_L$ any old 1-1 way (for each $L \in \mathcal{L}$.)

We leave it to the reader to verify that the induction hypotheses are preserved and that the claims concerning the result of the construction are true.

We claim that it follows that $B_n(T(M_n))$ is finite and that $B_{n+1}(T(M_n))$ is infinite. The latter claim is easily substantiated; indeed, by (ii) and (v) the formulas

$$(E_k z)(z < x_1 \wedge \ldots \wedge z < x_n)$$

are satisfied by distinct n-tuples of elements of M_n.

To see that $B_n(T(M_n))$ is finite, we note that any n-tuple has a unique configuration and hence satisfies a unique "configuration formula" φ. Now if ψ is any element of $B_n(T(M_n))$ then because of the theorem above, either

$$T(M_n) \vdash (x_1)\ldots(x_n)(\varphi \to \psi) \quad \text{or}$$

$$T(M_n) \vdash (x_1)\ldots(x_n)(\varphi \to \sim \psi) \;;$$

hence ψ is equivalent (in $T(M_n)$) to the disjunction of some subset of the set of configuration formulas. Hence $B_n(T(M_n))$ is finite.

It remains only to define $<$ so that (i)-(v) hold. This will be done by stages--at stage k we will define $\underline{a}_k$. We assume before we start stage $k+1$ that (ii) and (iii) hold when $i_1,\dots,i_{n+1}$ come from $1,\dots,k$ and also that infinitely many b's have been placed in none of $\underline{a}_1,\dots,\underline{a}_k$. At the end of stage $k+1$ (ii) and (iii) will continue to hold and in addition we will have met one of the requirements in (iv) or (v). In other words, we assume that we have listed all the requirements of (iv) and (v) so that each appears infinitely many times. Thus at stage $k+1$, we will try to do one of two things.

The first possibility is that we are presented with a finite subset B of $\{b_j\}_{j\in N}$ and we want $B\subseteq \underline{a}_{k+1}$; in this case, noting that by the induction hypothesis if $N\subseteq\{1,2,\dots,k\}$ with $\leq n$ elements then $(\bigcap_{x\in N}\underline{a}_x)\cap(\bigcap_{\substack{x\notin N\\ x<k}}\overline{\underline{a}_x})$ has infinitely many elements, for each $N\subseteq\{1,2,\dots,k\}$ with $\leq n$ elements we divide $(\bigcap_{x\in N}\underline{a}_x)\cap(\bigcap_{\substack{x\notin N\\ x<k}}\overline{\underline{a}_x})$ into two infinite pieces N_1 and N_2 , and we set $\underline{a}_{k+1}=(\bigcup_{N\in\mathfrak{n}}N_1)\cup B$ (where $\mathfrak{n}$ is the set of all subsets of $\{1,2,\dots,k\}$ with fewer than n elements.) Noting that for distinct N's in $\mathfrak{n}$ the sets above are disjoint, we can conclude that (ii) and (iii) continue to hold. Of course, we have succeeded in getting $B\subseteq\underline{a}_{k+1}$.

The second possibility is that we are presented with a case (v) requirement. (Note that if some of the a_i mentioned in the case (v) requirement have not yet been defined, we should just attack the next requirement.) We proceed as above to divide $(\bigcap_{x\in N}\underline{a}_x)\cap(\bigcap_{\substack{x\notin N\\ x<k}}\overline{\underline{a}_x})$ into two infinite pieces N_1 and N_2 for each $N\in\mathfrak{n}$ ($\mathfrak{n}$ as above) and we set $\underline{a}_{k+1}=(\bigcup_{N\in\mathfrak{n}}N_1)\cup(\bigcup_{L\in\mathcal{L}}B_L)$. Again (ii) and (iii) continue to hold, and again we have succeeded in meeting the requirement that presented itself at this stage.

This completes the construction of the systems M_n . I was unable to find in nature a system M_n with these properties for any $n\geq 2$.

After my talk at the Leeds Summer School, J. V. Howard pointed out examples for $n=2$ and $n=3$ which use infinitely many relations, but, except for that, do occur in nature. (There are other examples which involve an infinite number of relations.) For $n=2$, we define relations R_{ab} on the (rational) plane (a,b natural numbers) by $R_{ab}(x,y,z)$ if and only if $ax+by=(a+b)z$; it is then clear that $B_3(T(M))$ is infinite and that $B_2(T(M))$ is finite. For $n=3$, we define relations R_z on the (rational) complex plane (z (rational) complex) by $R_z(z_1,z_2,z_3,z_4)$ if and only if the cross ratio (z_1,z_2,z_3,z_4) is z ; since the cross ratio is invariant under linear transformations, and since any three distinct points can be carried to any other three distinct points by a linear transformation, it follows that $B_3(T(M))$ is finite and that $B_4(T(M))$ is infinite.

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