20

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TWO-DIMENSIONAL PARTIAL ORDERINGS: UNDECIDABILITY

ALFRED B. MANASTER AND JOSEPH G. ROSENSTEIN

In this paper we examine the class of two-dimensional partial orderings from the perspective of undecidability. We shall see that from this perspective the class of 2dpo's is more similar to the class of all partial orderings than to its one-dimensional subclass, the class of all linear orderings. More specifically, we shall describe an argument which lends itself to proofs of the following four results:

(A) the theory of 2dpo's is undecidable:

(B) the theory of 2dpo's is recursively inseparable from the set of sentences refutable in some finite 2dpo;

(C) there is a sentence which is true in some 2dpo but which has no recursive model;

(D) the theory of planar lattices is undecidable.

It is known that the theory of linear orderings is decidable (Laüchli and Leonard [4]). On the other hand, the theories of partial orderings and lattices were shown to be undecidable by Tarski [14], and that each of these theories is recursively inseparable from its finitely refutable statements was shown by Taitslin [13]. Thus, the complexity of the theories of partial orderings and lattices is, by (A), (B) and (D), already reflected in the 2dpo's and planar lattices.

As pointed out by J. Schmerl, bipartite graphs can be coded into 2dpo's, so that (A) and (B) could also be obtained by applying a Rabin-Scott style argument [9] to Rogers' result [11] that the theory of bipartite graphs is undecidable and to Lavrov's result [5] that the theory of bipartite graphs is recursively inseparable from the set of sentences refutable in some finite bipartite graph. (However, (C) and (D) do not seem to follow from this type of argument.)

Various smaller classes of partial orderings have decidable theories. For example, using a Feferman-Vaught technique [1], it is possible to deduce that the theory of products of two linear orderings is decidable from the fact that the theory of linear orderings is decidable [4]. It is also possible to show that the theory of weak orders is decidable. (A weak order is a partial ordering such that $(\forall x)(\forall y)(x < y \rightarrow (\forall z)(x < z \vee z < y))$; essentially this means the result of replacing each point in a linear order by a set of incomparable elements.) On the other hand, Schmerl [11] has recently shown that the class of partial orderings of width 2 (no 3 elements are pairwise incomparable) is undecidable, and has pointed out that since this is a definable subclass of the class of 2dpo's, this result also implies (A).

The existence of a consistent sentence with no recursive model was first shown by Mostowski [7], and Hanf [2] observed that this result is easily derived from his theorem on the existence of a finite set of tiles which can be used to tile the plane only nonrecursively. The proof of (C), which asserts the existence of such a sentence in the language of 2dpo's, uses tile models like those of Hanf.

A definition of planar lattices can be found in Platt [8]; planar lattices have also been investigated recently by Kelly and Rival [3]. We are indebted to G. McNulty for suggesting that our techniques could apply also to planar lattices.

To prove (A), (B) and (C), we will first define a class of structures called tile models and show that facts corresponding to (A), (B) and (C) are true of tile models. Then, by coding tile models into 2dpo's, we will obtain proofs of (A), (B) and (C). Our use of tile models is similar to the use of tiles in R. Robinson [10] and Hanf [2]; the proofs of (A) and (B) for tile models are in style of Büchi and the proof of (C) is similar to the proof of the main theorem of Hanf [2]. The coding of tile models into 2dpo's and the resulting proofs of (A), (B), and (C) are in the style of Rabin and Scott [9].

§1. **Tile models.** Intuitively, a "tile model" on b tiles consists of the set of lattice points of the upper half-plane, each point of which satisfies exactly one of b unary predicates. The predicates may be thought of as types of tiles (or, abusing language, as tiles) covering the squares whose lower-left corners are the lattice points of the model. Both predicates and tiles will be denoted $S_1, S_2, \ldots, S_b$. The description of each specific tile model will include conditions concerning the existence of certain tiles and restrictions on the possible tiles adjacent to any tile. Thus we may consider the tiles as having certain patterns along their edges which permit only certain tiles to fit together; the existence of a tile model is then equivalent to the existence of a specified type of tiling of the plane. (In order to formulate (B), we also need to allow finite tile models. Accordingly, we modify the description above to allow a tile model to consist of any "rectangular" set of lattice points of the upper half-plane; that is, any set of form $\{(i, j) \mid r < i < R \text{ and } u < j < U\}$ for some u, U, r, R satisfying $-\infty \leq r < R \leq \infty$ and $0 \leq u < U \leq \infty$.)

Since we want arbitrary interpretations of the language of tile models to look like tile models, it is convenient to add two binary predicates $\rightarrow$ and $\uparrow$ to the structure of tile models and corresponding symbols to the language of tile models. Intuitively, $x \rightarrow y$ (respectively, $x \uparrow y$) means that the lattice point y is immediately to the right of (respectively, above) the lattice point x.

Given a computation of a Turing machine, we will describe a tile model which represents that computation, with the tth row of the model representing the instantaneous description of the Turing machine at time t. Thus the point (c, t) will be covered by a tile which indicates the status of the cth cell of the tape after t steps of the computation. (The cth cell of the tape is the one which is c cells to the right—or $-c$ cells to the left, if $c < 0$—of the cell initially scanned by the Turing machine.) The type of the tile covering (c, t) will indicate

(a) the current symbol of the cth cell;

(b) whether or not the machine is currently scanning the cell and, if so, the state of the machine;

(d) the initial content of the cth cell and whether $c \geq 0$ or $c < 0$;

(e) for tiles off the bottom row, whether or not the machine was scanning the cell at the previous step; if so, the contents of the cell and the state of the machine at the previous step.

We could, of course, specify the tile model associated with any computation of a Turing machine more precisely and uniquely by fixing the method of determining which type of tile S_i encodes each specific set of data. Although we do not actually do so, we will still assume that some uniform method has been followed for encoding information into tiles. Note that, although an infinite number of types of tiles is needed to deal with all Turing machines, for any particular Turing machine only a finite number of types of tiles is required.

Now suppose we are given the Turing machine T_e, and suppose that the tiles $S_1, S_2, \ldots, S_b$ suffice to encode information about computations of T_e. We will soon define a statement $\mathscr{T}_e$ in the language of tile models on b tiles such that

(a) $\mathscr{T}_e$ is true of the tile model M_e of the computation of T_e which starts on the blank tape, and

(b) M_e is isomorphically embeddable in every model of $\mathscr{T}_e$.

Furthermore, if we let $H = \{j \mid 1 \leq j \leq b$ and S_j codes a halting configuration of $T_e\}$ and let $\mathscr{H}_e$ be $(\exists x)\bigvee_{j \in H} S_j(x)$, then $\mathscr{H}_e$ asserts that some point is covered by a tile which codes a halting configuration of T_e. (Note that the phrase "S_i codes a halting configuration of T_e" means that S_i indicates a cell with content σ being scanned by T_e in state q for a q and a σ on which T_e halts. When T_e halts, we consider the halting description to be continued at all later steps.) In particular $\mathscr{H}_e$ is true in M_e if and only if T_e halts when started on the blank tape.

Using the statements $\mathscr{T}_e$ and $\mathscr{H}_e$, we see that the theory of tile models (formulated in a language with infinitely many unary predicates) is undecidable. Indeed, if it were decidable, we would be able to decide, in particular, whether or not each $\mathscr{T}_e \Rightarrow \mathscr{H}_e$ is true in all tile models. However, if T_e does not halt when started on the blank tape, then $\mathscr{T}_e \Rightarrow \mathscr{H}_e$ is false in M_e; on the other hand, if T_e does halt, then $\mathscr{T}_e \Rightarrow \mathscr{H}_e$ is true in all tile models (and indeed in all appropriate structures) since every model of $\mathscr{T}_e$ contains a copy of M_e. Thus $\mathscr{T}_e \Rightarrow \mathscr{H}_e$ is true in all tile models if and only if T_e halts when started on the blank tape; hence any decision procedure for the theory of tile models would yield a decision procedure for the halting problem.

We now describe the statement $\mathscr{T}_e$ referred to above for (A). Later we will explain the slight modifications needed for (B) and (C). $\mathscr{T}_e$ is a conjunction of statements in the language of tile models. Each conjunct belongs to one of the following six classes according to the type of condition it expresses.

(1) *Growth.* For every point, there are unique points immediately to its right, immediately to its left, immediately above it, and, unless it is on the bottom row, immediately below it. (Note that, assuming that $B = \{i \leq b \mid S_i$ indicates a tile on the bottom row$\}$, the last statement can be expressed

$$(\forall x)[\bigvee_{i \notin B} S_i(x) \Rightarrow (\exists z)(z \uparrow x)] \wedge (\forall x)(\forall y)(\forall z)[y \uparrow x \wedge z \uparrow x \Rightarrow y = z].)$$

(3) *Partition.* Each point satisfies exactly one of the predicates S_i.

(4) *Initial description.* Some point is on the bottom row, and any point immediately to the right or left of a point on the bottom row is also on the bottom row. No point on the bottom row has a point immediately below it, and each point on the bottom row contains a blank. A unique point on the bottom row is scanned, and the tile on that point indicates that the Turing machine is in state q_1.

(5) *Transition rules.* If a point x is not on the bottom row, then it is covered by the same tile as the point y immediately below it unless any of the tiles covering y and the points immediately to its left and right indicate that the cell is being scanned. In addition, for each instruction $(q, \sigma, \sigma', q', D)$ of T_e and each tile S_i indicating that the current symbol is σ and that the cell is currently being scanned by the machine in state q (parts (a) and (b) of the tile indication), if the adjacent tiles to S_i are as pictured below for some symbols τ and τ'

S_n	S_j	S_c
$(\tau, -)\ S_m$	$(\sigma, q)\ S_i$	$(\tau', -)\ S_a$

then the current symbol and state indicators in the tiles S_n, S_j and S_c are

$(\tau, -)\ S_n$	$(\sigma', -)\ S_j$	$(\tau', q')\ S_c$
$(\tau, -)\ S_m$	$(\sigma, q)\ S_i$	$(\tau', -)\ S_a$

(In the figure we assumed $D = R$; the obvious analogue is intended for $D = L$.)

(6) *Halting rules.* For each halting configuration of T_e and each tile S_i indicating that halting configuration, the three tiles immediately above S_i and its immediate horizontal neighbors each indicate the same current symbol and state as the tiles immediately below them.

$(\tau, -)\ S_n$	$(\sigma, q)\ S_j$	$(\tau', -)\ S_c$
$(\tau, -)\ S_m$	$(\sigma, q)\ S_i$	$(\tau', -)\ S_a$

(Note that, since the (e) part of the indication of a tile may change when a halting configuration is first encountered, S_n, S_j and S_c may not be the same as S_m, S_i and S_a.)

The reader can verify that the statement $\mathcal{T}_e$ has the properties which were needed above to show that the theory of tile models is undecidable. To show that the theory of tile models is recursively inseparable from the set of statements which are finitely refutable requires only a few modifications. First recall that the set of Turing machines which halt when started on the blank tape is recursively inseparable from those which eventually restart on the blank tape—that is, which, after some time t, are again scanning a blank tape in the starting state q_1. Visualizing our tile whose width is suficient to encompass all cells used during the computation. We need, therefore, to alter $\mathscr{T}_e$, permitting this finite tile model to be a model of $\mathscr{T}_e$ without invalidating the argument based on $\mathscr{T}_e$. First, the growth requirement must be modified. Rather than asserting that immediate neighbors exist, we assert only that they are unique. Any point not on the bottom row does have a point immediately below it and, to guarantee rectangularity,

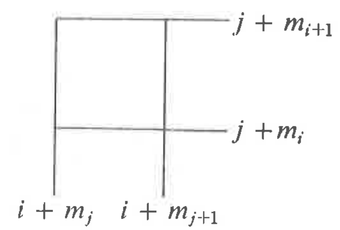
$$(\forall x)(\forall y)(\forall z)[(x \rightarrow y \wedge y \uparrow z) \Rightarrow (\exists z)(x \uparrow v \wedge v \rightarrow z)]$$

and three other statements asserting that any square with three corners in the model has all four corners in the model, must be included in $\mathscr{T}_e$. Secondly, the transition rules must be modified so that when the Turing machine moves right or left, there is a cell for it to move to. Finally, we add that any point whose tile indicates that the contents of its cell is not blank must have a tile immediately above it and that any point whose tile indicates that the cell is being scanned must have a point immediately above it unless the cell is being scanned in state q_1 and contains the blank symbol and the tile is not on the bottom row. Having revised $\mathscr{T}_e$, we need also revise the definition of M_e so that in all cases M_e is isomorphically embeddable in every model of $\mathscr{T}_e$; thus if T_e restarts on the blank tape, M_e will be the smallest finite model of T_e and otherwise, M_e, although infinite, may have finite width (in case the computation of T_e which starts on the blank tape is confined to a finite part of the tape). In any case, since $\mathscr{T}_e$ is true in M_e and M_e is isomorphically embeddable in every model of $\mathscr{T}_e$, we can use the argument above to conclude that the theory of tile models is recursively inseparable from the set of statements which are refuted in finite tile models.

The verification of (C) for tile models is as in Hanf [2]. Let A and B be recursively inseparable disjoint recursively enumerable sets, and assume that $0 \in A$. We will describe a Turing machine T which will be started on the right-most 1 of a tape which has infinitely many 1's and which is blank to the right of the cell initially scanned; if we define C by stipulating that $n \in C$ if and only if the nth cell to the left contains a 1, then the left-hand portion of the tape can be thought of as encoding the characteristic function of C. The Turing machine T enumerates the sets A and B and, when n is enumerated, halts if $n \in A$ and the $(-n)$th cell does not contain a 1 or if $n \in B$ and the $(-n)$th cell contains a 1. Thus T does not halt when started on C if and only if $A \subseteq C$ and $B \subseteq \bar{C}$. If we modify the statement $\mathscr{T}$ describing the action of T by saying that all bottom squares have 0's and 1's and that all bottom squares on the right have 0's, thus obtaining a statement $\mathscr{T}_0$, and take the conjunction with the statement $\neg\mathscr{H}$ asserting that no halting configuration occurs, then $\mathscr{T}_0 \wedge \neg\mathscr{H}$ has models but can have no recursive model—since given any recursive model, from the unique tile indicating a starting configuration we could recursively define the set C of the model and thereby obtain a recursive separation of A and B, contrary to hypothesis.

§2. **2dpo's associated with tile models.** We now associate with each tile model a 2dpo which will be defined by specifying a subset of the plane. Although the exact several 2dpo's which will be associated with a tile model share many common features.

We let $D_4, D_5, D_6, \ldots$ be a sequence of finite 2dpo's with the property that if $a \neq b$ then D_a is not embeddable in D_b. (Such a sequence is constructed in Lemma 4 of the previous paper.) Let $0 = m_0 < m_1 < m_2 < \cdots < 1/4$ be a sequence of rationals converging to 1/4 and let $m_{-i} = -m_i$ for each i, so that $-1/4 < \cdots < m_{-2} < m_{-1} < m_0$. A point is said to be near the lattice point (i, j) if it lies in the following rectangle



Similarly, a point is near the center of the square (i, j) if it lies in the rectangle

$$\{(x, y) \mid i + 1/2 + m_j < x < i + 1/2 + m_{j+1} \text{ and } j + 1/2 + m_i < y < j + 1/2 + m_{i+1}\}.$$

The 2dpo associated with a tile model will be a union of sets of points, called *boxes*, near lattice points and near centers of squares. The definitions of nearness above guarantee that given any two boxes either every point in one of the boxes is less than every point in the other box or every point in either box is incomparable with every point in the other box. This induces an ordering on the boxes and guarantees that the boxes of an associated 2dpo are ordered just as the points they are near.

The boxes placed near lattice points, called *corner boxes*, will always be isomorphic to one of $D_4, D_5, \ldots, D_{15}$, their purpose being simply to code the rectangularity of the tile model. For (A) and (B) the pattern of the order types of the corner boxes will be simply

8	9	8	9	8	9	8	9
10	11	10	11	10	11	10	11
8	9	8	9	8	9	8	9
10	11	10	11	10	11	10	11
8	9	8	9	(8)	9	8	9

where n represents the order type of D_n and the type of the corner box near (0, 0) is circled. For (C) the pattern of the order types of the corner boxes will be

12	13	4	5	8	9	8	9	8	9	8	9	8
14	15	14	7	6	11	10	11	10	11	10	11	10
12	13	12	13	4	5	8	9	8	9	8	9	8

Given a tile model, corner boxes will be placed in the associated 2dpo near each point (i, j) of the tile model and also near each point $(i + 1, j + 1)$ for which at least one of $(i + 1, j)$, $(i, j + 1)$, (i, j) is in the tile model. No other lattice points will have boxes near them in the associated tile model.

The boxes placed near center points, called *center boxes*, will always be isomorphic to one of $D_{16}, D_{17}, \ldots, D_{15+b}$, their purpose being to code the tile structure of the tile model on the tiles $S_1, S_2, \ldots, S_b$. Indeed, if the point (i, j) of the tile model is covered by the tile S_a then the center box near the center of the square (i, j) has order type D_{15+a}.

This completes the description of the 2dpo associated with a tile model. With a specified pattern of corner boxes, it is unique up to isomorphism.

It is perhaps possible to associate simpler 2dpo's with tile models and still be able to prove (A) and (B). However to prove (C) we make critical use of the fact that no D_a is embeddable in any D_b. To see how this assumption is used, imagine trying to decode some information from a recursive 2dpo. You would first have to find the place where that information is stored; that is, you would have to find elements of the 2dpo which have a certain configuration. Suppose you find such elements—perhaps they are not the only ones with the desired configuration, perhaps they are a part of a larger, similar, configuration denoting another address. You cannot answer these questions in general; but if the configurations sought are D_a's, then the doubts can be resolved. Thus, for example, if a recursive 2dpo has the second pattern of corner boxes, then, starting from the corner boxes of types 4 and 5 on the bottom row, it is possible to find, recursively, the corner boxes of types 4 and 7 along the critical northwesterly diagonal.

We now must verify that a number of concepts concerning 2dpo's associated with tile models on b tiles are definable in the language of 2dpo's, which contains only $\leq$ as a relation symbol and no other extra-logical symbols. (By "definable" we mean that there is a formula in the appropriate number of free variables which defines the described relation in every 2dpo associated with a tile model on b tiles.)

(1) $\{x_1, x_2, \ldots, x_{f(i)}\}$ constitutes a box.

Here $4 \leq i \leq 15 + b$ and $f(i)$ is the number of elements of D_i. ($f(i) = 2i + 1$ for the 2dpo D_i of [6].) This may be expressed by a formula which asserts, first, that some permutation of $\{x_1, x_2, \ldots, x_{f(i)}\}$ is isomorphic to D_i as a partial ordering and, secondly, that any element not among $\{x_1, x_2, \ldots, x_{f(i)}\}$ stands in the same relation to all of them. (The second assertion is required since, for example, D_i is also realized in associated 2dpo's by selecting at most one point from each of a set of corner boxes of the 2dpo.) The formula expressing this property is denoted $D_i(x_1, x_2, \ldots, x_{f(i)})$ or simply D_i.

(2) x_1 and x_2 are in the same box.

The required formula, $\bigvee_{i=4}^{15+b}(\exists x_3) \cdots (\exists x_{f(i)})D_i$, is denoted $x_1 \approx x_2$.

(3) x is in a box of type i.

(4) x is in a corner box.

(5) x is in a center box.

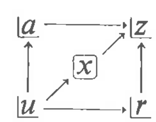
(6) x and y are in corner boxes and the corner box of y is immediately to the right of the corner box of x.

Using our conventions about corner boxes, this can be expressed by the formula

$$\bigvee_{4\le i\le 15}[\bar{x}=i\wedge \bigvee_{\substack{4\le j\le 15\\ j+i\ \text{odd}}} y=j]\wedge x<y\wedge (\forall z)[x<z<y\Rightarrow (x\approx z\vee z\approx y)].$$

This formula is abbreviated $\lfloor\underline{x}\rightarrow\lfloor\underline{y}$. A similar formula defines $\lfloor\underline{x}\uparrow\lfloor\underline{y}$. Corresponding formulas for center boxes are defined similarly and denoted $\boxed{x}\rightarrow\boxed{y}$ and $\boxed{x}\uparrow\boxed{y}$. Formulas which express similar relationships between elements of corner boxes and elements of center boxes may also be defined and are denoted $\lfloor\underline{x}\nearrow\boxed{y}$ and $\boxed{x}\nearrow\lfloor\underline{y}$.

These formulas enable us to express certain structural conditions which are satisfied by all 2dpo's associated with tile models on b tiles; the conjunction of the conditions stated below is denoted G_b. First, G_b asserts that $\le$ is a partial ordering. Second, G_b asserts $(\forall x)\bigvee_{i=4}^{15+b}\bar{x}=i$, which guarantees that any model of G_b is a union of boxes of types $D_4, D_5, \ldots, D_{15+b}$ and that the ordering of the model induces an ordering on the boxes. Next, G_b says that given any corner box $\lfloor\underline{u}$, there is at most one corner box $\lfloor\underline{a}$ immediately above it, at most one corner box $\lfloor\underline{b}$ immediately below it, at most one corner box $\lfloor\underline{r}$ immediately to its right, at most one corner box $\lfloor\underline{l}$ immediately to its left, at most one center box $\boxed{x}$ such that $\lfloor\underline{u}\nearrow\boxed{x}$, and at most one center box $\boxed{y}$ such that $\boxed{y}\nearrow\lfloor\underline{u}$. Furthermore, if $\lfloor\underline{u}$ is of type 10, for example, and if any of $\lfloor\underline{a}$, $\lfloor\underline{b}$, $\lfloor\underline{r}$ or $\lfloor\underline{l}$ exist, then they must be of types 8, 8, 11, 11 respectively. (Similarly for other possible types of $\lfloor\underline{u}$; for (C) there are 11 such statements, in some of which the types of the corner boxes are not completely determined by the type of $\lfloor\underline{u}$.) Next G_b says that given any three corner boxes which form three vertices of a square (there are four such configurations) there is a corner box which completes the square and a center box which lies within the square so that the five boxes are in the following configuration



Furthermore, any center box lies within such a square of corner boxes. Finally, there is a corner box of type 8 (or, of type 4, for (C)) which has no corner box immediately below it.

If P is a model of G_b and $\lfloor\underline{u}$ is any corner box of type 8 (respectively, 4) which has no corner box immediately below it and we let P^* consist of those elements of P which can be "reached" from $\lfloor\underline{u}$ using arrows in a finite number of steps, then P^* is also a model of G_b. That is to say, P^* decomposes into boxes, the corner boxes of P^* will form a rectangular set with bottom row containing $\lfloor\underline{u}$, the types of the corner boxes of P^* will form a rectangular part of the required pattern, and the center boxes of P^* will be properly located, although, in general, the information stored in them is gibberish. Our next task, therefore, is to degibberize the center boxes.

each formula φ in the language of tile models determines a formula $\hat{\varphi}$ in the language of partial orderings such that φ is satisfied in a tile model by certain points if and only if $\hat{\varphi}$ is satisfied in the associated 2dpo by elements of the center boxes near those points. The inductive passage from φ to $\hat{\varphi}$ begins by replacing each formula $S_i(x)$ by $\bar{x} = 15 + i$, each $x = y$ by $x \approx y$, each $x \to y$ by $\boxed{x} \to \boxed{y}$, and each $x \uparrow y$ by $\boxed{x} \uparrow \boxed{y}$; the inductive steps are straightforward, quantifiers being relativized to elements of center boxes.

Using this interpretation, we show that (A) and (B) hold. Let $\mathscr{T}^e$ be the conjunction of $\hat{\mathscr{T}}_e$, G_b (where b is determined by e), and a statement which asserts that the corner box immediately below the center box of the initial tile is of type 8. (Thus if i is the type of the tile indicating a cell being scanned in the bottom row, the statement is

$$(\exists x)[\bar{x} = 15 + i \wedge (\forall y)(\bar{y} = 15 + i \Rightarrow y \approx x)$$
$$\wedge (\exists u)[\lfloor u \nearrow \boxed{x} \wedge \bar{u} = 8 \wedge \neg(\exists v)(\lfloor v \uparrow \lfloor u)]].)$$

Let $\mathscr{H}^e$ be the statement $\hat{\mathscr{H}}_e$.

The 2dpo P^e associated with the tile model M_e of the computation of T_e on the blank tape satisfies $\mathscr{T}^e$ and also satisfies $\mathscr{H}^e$ just in case the computation halts. Thus, if the Turning machine T_e does not halt when started on the blank tape, then $\mathscr{T}^e \Rightarrow \mathscr{H}^e$ is false in P^e; if moreover, T_e eventually restarts on the blank tape, then $\mathscr{T}^e \Rightarrow \mathscr{H}^e$ is refuted in the finite model P^e.

To complete the proofs of (A) and (B), we need only show that if T_e halts when started on the blank tape, then $\mathscr{T}^e \Rightarrow \mathscr{H}^e$ is valid. Now any model P of $\mathscr{T}^e$ has a unique center box indicating an initial tile; the submodel P^* of P, generated by the corner box of type 8 immediately below it, is isomorphic to P^e. Hence $\mathscr{T}^e \Rightarrow \mathscr{H}^e$ is valid.

Thus we may conclude that (A) the theory of 2dpo's is undecidable, and (B) the theory of 2dpo's is recursively inseparable from the set of statements which are refuted in finite 2dpo's.

§4. A statement with 2dpo models but no recursive models. To prove (C) we must construct a statement $\mathscr{A}$ which has models which are 2dpo's, but so that none of its models are recursive. Recall that the statements $\mathscr{T}_0$ and $\neg\mathscr{H}$, described at the end of §1, have tile models but no recursive models. Let G be a suitable modification of that G_b (as discussed in §2) for the b appropriate to the Turing machine used in the definition of $\mathscr{T}_0$. The statement $\mathscr{A}$ will be the conjunction of $\mathscr{T}_0$, $\neg\mathscr{H}$, G, and two additional statements. The first additional statement asserts that the corner box immediately below the unique center box whose type codes both "bottom row" and "currently scanned" is of type 4. The second additional statement is a technicality which will simplify the proof and will be described in more detail later.

It is clear that $\mathscr{A}$ has a model which is a 2dpo, and, in fact, that $\mathscr{A}$ has a continuum of different countable models which are 2dpo's; indeed, given any set C which separates the recursively inseparable sets A and B, one model of $\mathscr{A}$ is that 2dpo which is associated with the tile model of the computation of T starting on the cursive in a recursive 2dpo. We will use the pairwise nonembeddability of the D_n's and the special corner boxes on the crucial diagonal—northwesterly from the initial corner box of type 4—to recover, recursively in any model of $\mathscr{A}$, the initial tape of the computation. Thus any recursive model of $\mathscr{A}$ would give a recursive separation of A and B; hence there can be no recursive model of $\mathscr{A}$.

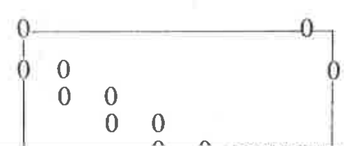
Consider, for example, a D_4 box and its adjacent D_5 box. We want to find, recursively, the D_6 box adjacent to the D_4 box. While there is only one D_6 box greater than the D_4 box, there are many sets of points in any model of $\mathscr{A}$ which are greater than the D_4 box and isomorphic to D_6. (For example, take one point from each of a suitable set of corner boxes.) However, there is only one set of D_6 points which is isomorphic to D_6 and is such that every point in the set is both greater than the given D_4 box and incomparable with the adjacent D_5 box. Thus, if P is a recursive model of $\mathscr{A}$, then we can, starting with the initial D_4 box and its adjacent D_5 box, successively find in a recursive fashion the D_6 box above the D_4 box, the D_7 box to the left of the D_6 box, the D_5 box above the D_7 box, the D_4 box to the left of the D_5 box, etc. The easiest way to ensure that the uniqueness of these boxes is a consequence of $\mathscr{A}$ is to conjoin another statement to $\mathscr{A}$. This statement is itself a conjunction of four statements, only one of which is presented here:

$$(\forall x)(\forall y)\{\bar{x} = 4 \wedge \bar{y} = 6 \wedge \underline{\lfloor x} \uparrow \underline{\lfloor y} \Rightarrow$$
$$(\forall z)[\{(\exists x_2) \cdots (\exists x_{f(7)}) D_7(z, x_2, \ldots, x_{f(7)}) \wedge z < y \wedge z \mid x\} \Rightarrow \underline{\lfloor z} \rightarrow \underline{\lfloor y}]\}.$$

Now suppose that P is a recursive model of $\mathscr{A}$ and that P^* represents the computation of the Turing machine T beginning on a tape which encodes a set C. We show that C is recursive. Indeed, to determine whether or not $n \in C$, starting with the (unique) D_4 box which is a bottom box of P and the D_5 box on its immediate right, proceed, as in the paragraph above, to find the nth corner box D on the crucial diagonal. (If n is even, the nth corner box on the crucial diagonal will be a D_4 box; if n is odd, it will be a D_7 box.) Find the unique center box E which is above D but is incomparable with the D_5 and D_6 corner boxes which are immediately above and to the right of D (not necessarily respectively). Since the center boxes code the initial contents of the cells, E codes the initial contents of the $(-n)$th cell of the tape. Thus from E we can tell whether or not $n \in C$. Hence C is recursive, a contradiction. Therefore, $\mathscr{A}$ can have no recursive models, proving (C).

§5. The undecidability of the theory of planar lattices. It is evident that if all of the 2dpo's described above were also lattices then we would also have results (A), (B) and (C) for planar lattices. Although they are not lattices, they are easily modified to become lattices.

(1) Instead of using D_n as pictured on the left (for $n = 8$) use the lattice L_n pictured on the right.



(The proof that L_a is not embeddable in L_b if $a \neq b$ is as in Lemma 4 of [6].)

(2) Place center boxes so that any two in the same row or column are incomparable. This can be arranged by redefining the rectangle "near" the center of the square (i, j) to be

$$\{(x, y) \mid i + 1/2 + m_{-j} < x < i + 1/2 + m_{-j+1} \text{ and } j + 1/2 + m_{-i} < y < j + 1/2 + m_{-i+1}\}.$$

With these stipulations, all of our 2dpo's are planar lattices, so conclusions (A), (B) and (C) are correct for planar lattices.

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UNIVERSITY OF CALIFORNIA, SAN DIEGO
LA JOLLA, CA 92037

RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08903

INSTITUTE OF ADVANCED STUDY
PRINCETON, NJ 08540