Discrete Mathematics and the Chinese Postman Problem

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Principles and Standards for School Mathematics of the National Council of Teachers of Mathematics (NCTM) recommends that "discrete mathematics should be an integral part of the school mathematics curriculum" (NCTM, 2000, p. 31), and recognizes three areas of discrete mathematics as important at all primary and secondary grade levels – combinatorics (that is, systematic listing and counting), iteration and recursion (that is, modeling change discretely), and vertex-edge graphs.

In this article, we present one practical problem involving vertex-edge graphs and use it to illustrate why discrete mathematics should be an integral part of the school mathematics curriculum.

The recent two-volume publication by NCTM, *Navigating Through Discrete Mathematics,* by Valerie DeBellis, Eric Hart, Margaret Kenney, and Joseph Rosenstein (2008 and 2009), describe in detail how these three areas can be implemented in primary and secondary classrooms. Two important collections of articles about discrete mathematics in primary and secondary education are *Discrete Mathematics in the Schools* (1997) edited by Joseph Rosenstein, Deborah Franzblau, and Fred Roberts, and *Discrete Mathematics Across the Curriculum K-12*, the 1991 NCTM Yearbook, edited by Margaret Kenney and Christopher Hirsch.

Discrete mathematics is not a well-defined area of mathematics, like algebra or geometry, and so there are many definitions of discrete mathematics and many lists of topics that it includes. The question, What is Discrete Mathematics?, is discussed in detail in two articles by Steven Maurer and Joseph Rosenstein in *Discrete Mathematics in the Schools*, and in Rosenstein's 2005 article "Discrete Mathematics in 21st Century Education: An Opportunity to Retreat From the Rush to Calculus." Rosenstein's article in *Discrete Mathematics in the Schools* provides recommendations and classroom activities at each grade level for all areas of discrete mathematics.

Vertex-edge graphs can be used to represent a number of problems that arise in the real world and, because the discussion and solution of these problems do not require many mathematical prerequisites, these problems can be introduced and fruitfully discussed in secondary and even primary schools.

For example, imagine a person whose job is to deliver the mail (a "postman") to all the houses in a district of the city. Can the postman construct a route which will allow all of the mail to be delivered without repeating any streets? For example, if the map of the district is that of Figure 1, and the postman starts at P, a route that travels on all the streets without any repetitions is indicated in Figure 2.

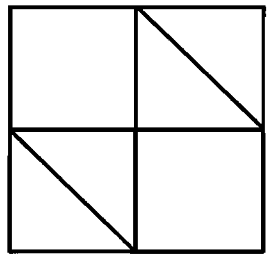


Figure 1

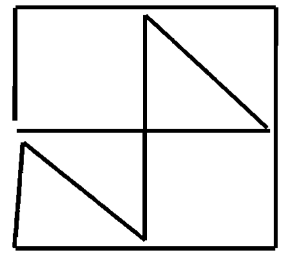


Figure 2

(Two routes beginning at P are actually pictured in Figure 2, one which begins by going to the right and the other which begins by going upward; of course, each of these two routes is the reverse of the other.)

This problem is equivalent to the problem of whether the entire picture in Figure 1 can be drawn starting at P without removing pencil from paper and without repeating any part, that is, if it can be drawn in one continuous line. Young children can be provided many such pictures and asked to draw them. A favorite example is the "house" in Figure 3. This picture can only be drawn in a single line without repetitions if you start at the bottom left corner and end at the bottom right corner (or the reverse).

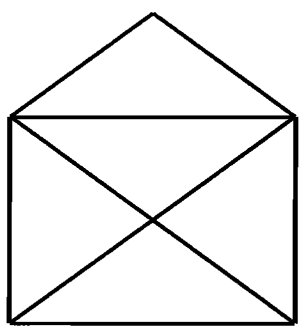


Figure 3

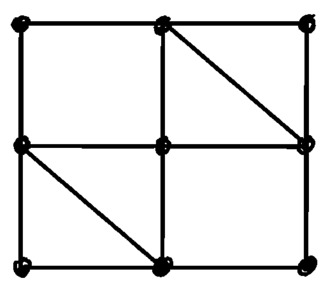
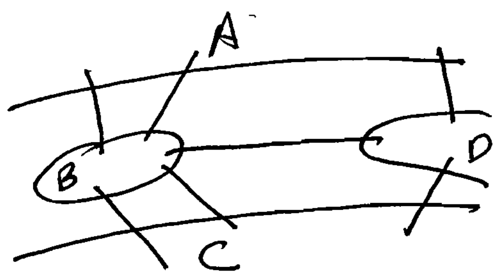


Figure 4

Such problems can be represented using dots (called "vertices") for the intersections and lines (called "edges") for the streets. Each street joins two intersections, and so each edge joins two vertices. Figure 4 contains the vertex-edge graph that corresponds to the map in Figure 1. Vertex-edge graphs were originally introduced by the great 18th century mathematician Leonhard Euler to solve a problem of this type, known as the Konigsberg Bridge Problem. In his time, the city of Konigsberg was located on both sides of the Pregel River and included two islands; these four regions were connected by seven bridges (see Figure 5).



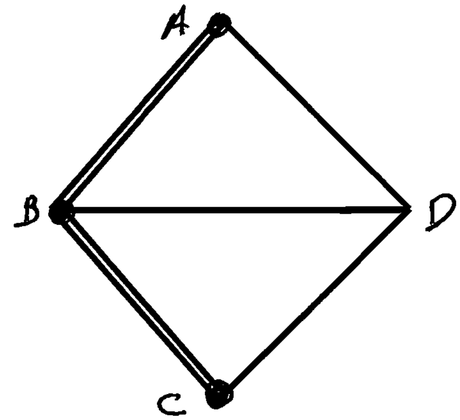


Figure 5

Figure 6

The question was whether one could go for a walk that included crossing each bridge exactly one and that ended up at the starting point. Figure 6 contains a vertex-edge graph that describes this situation; in this graph the four vertices represent the four regions and the seven edges represent the seven bridges. In terms of vertex-edge graphs, the question that Euler was asked to decide was whether there is a path that includes each of the seven edges exactly once and that ends where it begins.

The key sequence of observations that children and adults can be led to discover about paths in vertex-edge graphs is:

1. if a path goes through a vertex then it goes in along one edge and out along a different edge;
2. if a path goes through a vertex a number of times without repeating any edges, then it has traveled along an even number of edges that end at that vertex;
3. if a path includes each edge of a graph, but doesn't repeat any edge, then at each vertex (except at the first and last vertex) there are an even number of edges.

If we define the "degree" of a vertex to be the number of edges that end at the vertex, two conclusions can be drawn if there is a path that includes each edge of the graph exactly once:

4. if the path ends where it begins, then all of the vertices must have even degree, as in the graph in Figure 4.
5. if the path does not end where it begins, then all of the vertices except two must have even degree, and the path must begin and end at the two vertices of odd degree, as in the "house" in Figure 3.

These observations allow us to look at a picture and determine very quickly that it *cannot* be drawn with one continuous line – if it has more than two vertices of odd degree, then such a drawing does not exist. However, these observations do not guarantee that the picture can actually be drawn if there no vertices of odd degree or two vertices of odd degree. But Euler proved that this is indeed true. If a vertex-edge graph has 0 or 2 vertices of odd degree, then there is a path in the graph that includes each edge exactly once.

A path that includes each edge exactly once is called an "Euler path"; if the path ends where it begins, then it is called an "Euler circuit." Finding an Euler path or an Euler circuit in a graph can be challenging. You can construct interesting examples by starting with a cycle, as in Figure 7, and then connecting each vertex to two additional vertices, as in Figure 8. Each vertex in this vertex-edge graph has degree 4, so there must be an Euler circuit; can you find one?

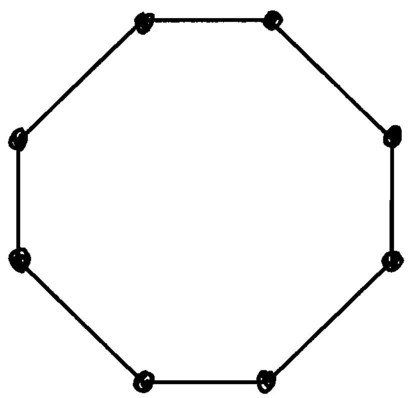


Figure 7

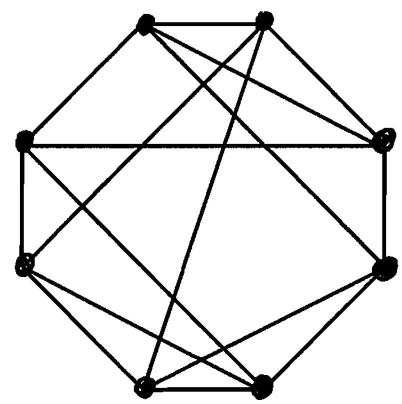


Figure 8

You can do this type of problem in a classroom setting by having 10 to 15 students stand in a circle to represent the vertices. Each student is given a string (about 2 meters long) and must give one end of the string to another student, but not one of those he or she is standing next to; each student must receive one string in addition to the one that he or she started with. Once all of the students have two strings they put their strings on the floor and stand on the ends of their strings. The students then hold hands with the two students who are next to them. They have now formed a vertex-edge graph where each vertex has degree 4 – the edges are strings and pairs of hands that are joined. One of the other students now tries to create an Euler circuit – by starting at some vertex, walking to another vertex, then to another vertex, then to another vertex, etc. As the student walks along each edge, the edge is removed so that it will not be used again; the strings can actually be removed and the pairs of hands can be separated. (Before any strings are removed, it is wise to record the graph on paper so that the graph can be reconstructed if the first attempt to create an Euler circuit is unsuccessful.) The students will quickly discover that a good strategy for finding an Euler circuit is to walk on the internal edges for as long as possible and then complete the Euler circuit by walking around the cycle on the outside of the graph. This activity is an example of what we call a "human graph activity," where the vertices of the graph are students.

Digression. We noted above that a vertex-edge graph has an Euler path or an Euler circuit if it has 0 or 2 vertices of odd degree. A natural question that students may ask is, What happens if a vertex-edge graph has 1 vertex of odd degree? You might challenge your students to draw a vertex-edge graph with 4 vertices of degree 2 and 1 vertex of degree 3, or more interestingly, 3 vertices of degree 3. After five or ten minutes, ask them whether anyone has succeeded in finding such a graph; none of them will have found an example. Give them a few more minutes and ask them to add together the degrees of the all the vertices of one of the graphs that they have drawn. Ask them how many have totals that are even, and how many have totals that are odd. What will happen is that all of the totals are even. (If someone claims an odd total, have him or her draw the graph on the board and discuss the graph.) Ask the students why the total of the degrees of the vertices is always even. If no one has an answer, ask them what they notice about the number that is half the total. Someone will notice that this is the number of edges in the graph. How can we explain why this happens? When you add together the degrees of all the vertices in a graph, each edge is counted twice, once at each end – so the total of the degrees must be twice the number of edges. But if there are an odd number of vertices of odd degree, then the total will be odd, so a vertex-edge graph cannot have an odd number of vertices of odd degree. End of Digression.

We return now to the topic of this article. Although we have come to a nice conclusion, most vertex-edge graphs, unfortunately, do not have Euler paths or circuits, and most postmen will not be able to construct routes with no repeated streets. Does that mean that the mail will not be delivered? We hope not. Instead we ask the more interesting question, What is the best route for the postman? Consider for example the vertex-edge graph in Figure 9.

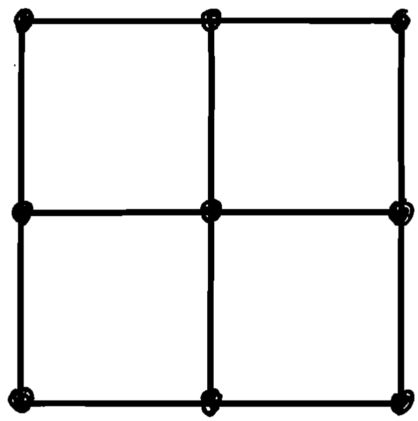


Figure 9

We know that it has no Euler path or circuit since it has 4 vertices of degree 3. It has 12 edges so we know that the best route for the postman must have more than 12 edges. So the question we might ask is, What is the smallest number of edges in a path or a circuit that includes all of the edges?

One way of answering this question is by adding edges to the graph in order to force all of the vertices to become even – a process called "balancing" or "Eulerizing" the graph. Thus if we add two edges to Figure 9, as indicated in Figure 10, we now have a graph that has only 2 odd vertices and therefore has an Euler path that begins at one of the vertices of degree 3 and ends at the other. Note that the two added edges duplicate edges that already exist, so that in Figure 10 there are now two edges joining vertices A and B and two edges joining edges B and C.

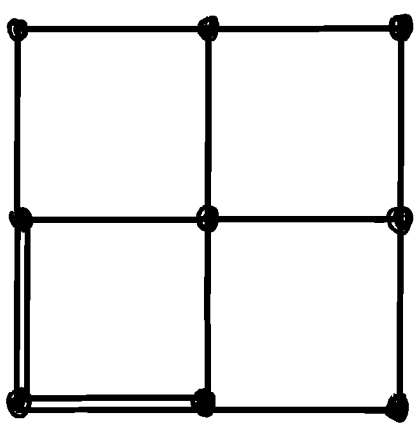


Figure 10

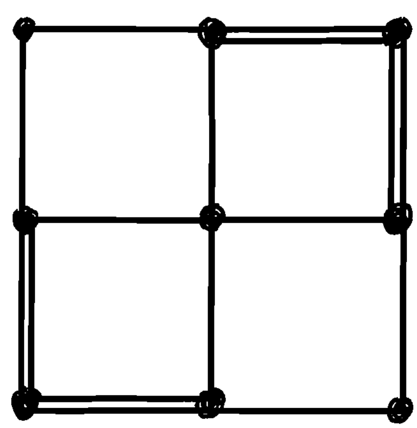


Figure 11

If we add two more edges, as in Figure 11, we now have a graph that has no odd vertices and therefore has an Euler circuit. Thus, we can find a route for the postman in the vertex-edge graph of Figure 9 that ends where it begins and includes only four extra edges. A drawing of this route appears in Figure 12. One way of describing the route is by labeling the vertices, as in Figure 13; then the route in Figure 12 can be described as A-B-C-F-E-D-G-H-I-F-C-B-E-H-G-D-A.

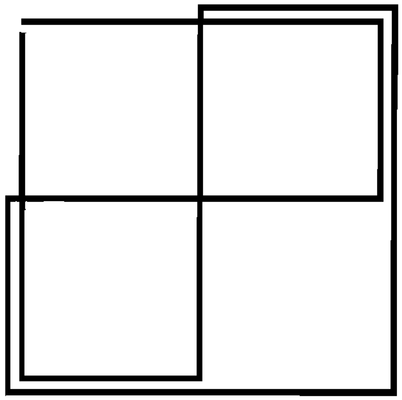


Figure 12

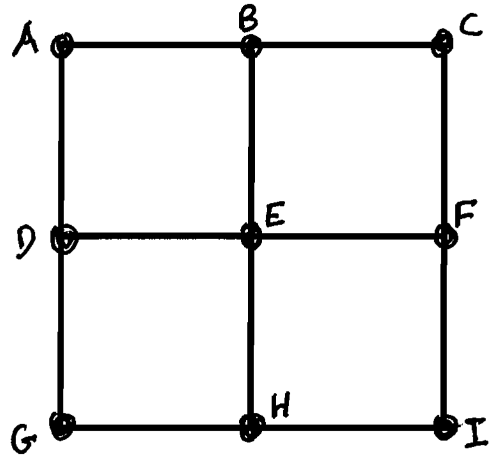


Figure 13

The question of how many extra edges are needed in a graph to ensure that there is a circuit that includes all edges was studied by the Chinese mathematician Guan Mei-gu in the 1970s, and in his honor is known as the Chinese postman problem.

An important variation of this problem, one that makes it more realistic, is one that recognizes that not all added edges are equal, that the streets that are repeated should be ones which are shorter. This involves the concept of a "weighted graph," where each edge is assigned a weight. In this situation, the weight would be distance, but in other applications, the weight might reflect cost or time. In this context, the postman's problem would be to find a circuit in the graph that includes all of the edges and has the least total weight. For example, the edges of the graph might have the weights indicated in Figure 14. If the postman's route repeated the same four edges as in Figure 11, that would add 10 to the total (see Figure 14).

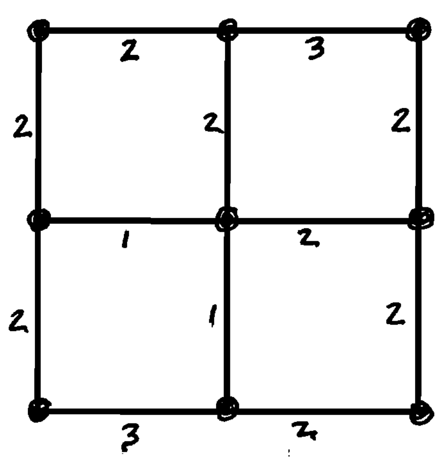
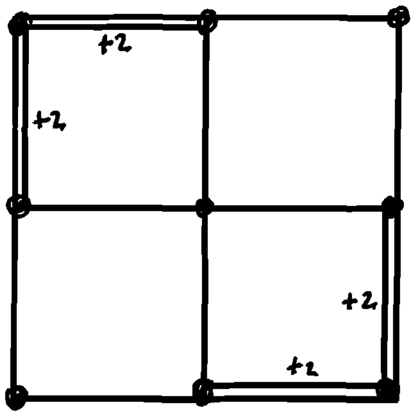


Figure 14

Can we do better? Yes, we can instead add the four edges in Figure 15, that would add only 8 to the total. Are there any other possibilities? Yes, we could instead add the four edges in Figure 16, and that would add only 6 to the total. It might even be possible in some situations that adding more than the minimum number of additional edges would add a smaller weight to the total.



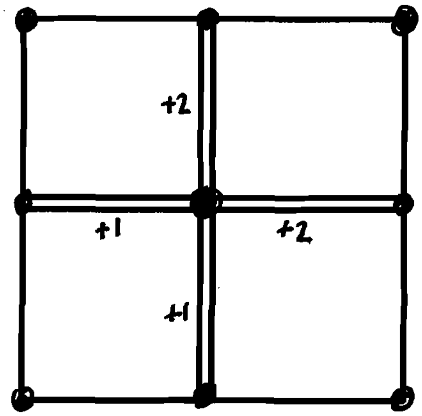
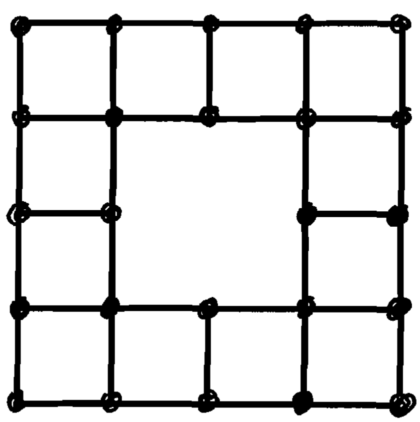


Figure 15 Figure 16

A final example. The graph in Figure 17 has 24 vertices, 16 of degree 3 and 8 of degree 4. What is the smallest number of edges in a circuit that includes all 36 edges? If we add weights to the 36 edges as indicated in Figure 18 – where half the edges have weight 1 and half have weight 2, so that the total weight is 54 – what is the smallest total weight of the edges in a circuit that includes all 36 edges?



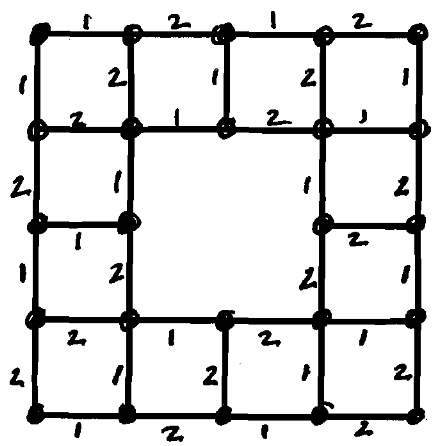


Figure 17 Figure 18

Another final example. How many different ways are there of Eulerizing the vertex-edge graph in Figure 9 by adding four extra edges?

Conclusion

In this article, we have presented one class of practical problems involving vertex-edge graphs. From considering this type of situation, we can see a number of reasons why topics of discrete mathematics should be, as recommended by the NCTM, an integral part of the school mathematics curriculum.

First, working on problems like these help students develop their reasoning and problem-solving skills.

Second, problems like these are accessible to primary and secondary students. In order to discuss these problems, the students do not require a great deal of prior information, and the solutions of these problems do not require a great deal of computation, so that the focus can be on reasoning and problem-solving.

Third, discrete mathematics includes a variety of mathematical problems that arise in everyday life, so that students can come to understand how mathematics is used in everyday situations.

Fourth, all students find these problems engaging, including those who have been less successful at traditional mathematical topics, and each type of problem situation can provide challenging problems to mathematically talented students.

Fifth, because they are engaged in the learning, students come to understand that mathematics is something that you *do* and not just something you *learn*. Students explore a situation, ask questions, make conjectures, test their conjectures, and explain their reasoning, just as mathematicians do.

There are indeed many good reasons for including study of such problem situations in the primary and secondary mathematics curriculum.

Could these outcomes be achieved using more traditional topics in mathematics? In theory, they could be. However, in practice, our curricula and classrooms are so focused on providing students with information, on teaching students computation, and on preparing students for assessments, that these outcomes are not the focus of our instruction. Moreover, our teachers are not prepared to teach traditional topics in mathematics in a way that emphasizes such outcomes.

Are these outcomes actually achieved by using topics in discrete mathematics? This question is, unfortunately, difficult to answer, because no systematic research has been conducted that would help answer the question about the effects of discrete mathematics on primary and secondary students. However, we have a good deal of information about the effects on teachers. Over the past fifteen years, we have conducted over 50 institutes for elementary school and middle school teachers, that is, teachers of students age 5 to 14. Each institute involved between eight and fifteen full days of instruction on topics dealing with discrete mathematics. Over 1000 teachers have been involved in these institutes, which generally took place over two summers, with follow-up sessions during the intervening school year.

From the evaluation forms that the teachers completed and from the anecdotes that they related to us about what they did subsequently in their classrooms, it is clear that the five objectives above were achieved for the teachers. During their times at the institute, they were *doing* mathematics, they were solving problems, they were engaged in the mathematics, they saw the broad scope of mathematical applications, and they saw the value of mathematics in dealing with everyday problems. Many of these teachers had never before dealt with or solved a mathematical problem – their past exposure to mathematics was to see a numerical or algebraic computation and do many similar examples, and that's how they taught their students. Many of these teachers were afraid of mathematics, but in the institute they experienced success in mathematics. Now that they had a better understanding of mathematics, they could convey it to their students; now that they had achieved success themselves, they could expect it of all of their students.

Based on the workshops at these institutes, Rosenstein and DeBellis have developed text materials for a college course in discrete mathematics for prospective elementary and middle school teachers, *Making math engaging: Discrete mathematics for prospective K-8 teachers*. A more extensive discussion of discrete mathematics in the United States and its history can be found in DeBellis and Rosenstein (2004).

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