Questions raised by Quora readers, related to counting sets, each followed by my response:

How many onto functions from the set {a, b, c, d} to the set {1, 2, 3} are possible?

Any such function must map two elements of the initial set {a,b,c,d} to one element of the terminal set {1,2,3}.

Let's first see how many functions map two elements of the initial set to 3. I could choose any two of the elements of the initial set. There are "4 choose 2" ways of doing that, that is, 6 ways. The remaining two elements of the initial set have to map onto {1,2}, and there are two ways of doing that. So, by the Multiplication Principle of Counting, there are 6x2=12 functions that map the initial set onto the terminal set, and that map two elements of the initial set to 3.

By symmetry, there are 12 onto functions that map two elements to 2, and there are 12 onto functions that map two elements to 1.

Altogether, by the Addition Principle of Counting, there are 12+12+12=36 possible functions that map the initial set onto the terminal set.

If a set A is such that A= {1,2,3,5,7} find the power set P(A)?

As all previous responders have noted, the power set of A has 32 elements, that is, A has 32 subsets. I will address the question of how you can list systematically this list of 32 subsets.

Think of any particular subset B of A as determined by the answers to five questions: Is 1 in B? Is 2 in B? Is 3 in B? Is 5 in B? Is 7 in B? If, for example, the answers to those questions are Y, Y, N, N, Y (where Y is yes and N is no), then B = {1,2,7}. So subsets of A correspond to sequences of five N's and Y's.

Make sure that you understand why there are exactly 32 such sequences.

Now you can make a list of all sequences of five N's and Y's — and using that list make a list of all subsets of A.

How can you do that systematically? Use the alphabet! — that is, make an alphabetical list of all sequences of five N's and Y's.

Here is how that list would begin

NNNNN, NNNNY, NNNYN, NNNYY, NNYNN, NNYNY, NNYYN, NNYYY, …

You should complete that list of 32 sequences, in alphabetical order, and then use that list to find the 32 subsets of A that correspond to the 32 sequences.

How do I prove that the minimum edges of a graph must be n-k, where n is the number of nodes and k is the number of connected components?

If a connected component of a graph has A nodes, then it must contain at least A-1 edges. This you can prove by induction on A.

Suppose now that a graph has k connected components, with A1, A2, … , Ak nodes, respectively, and a total of n nodes. Then by the statement above, the components have at least (A1 - 1) + (A2 - 1) + … + (Ak - 1) elements, which is exactly n-k.

It's dark. You have ten grey socks and ten blue socks you want to put into pairs. All socks are exactly the same except for their colour. How many socks would you need to take with you to ensure you had at least a pair?

Any two socks form a pair of socks, so you would only need to take two socks to ensure that you have a pair. Now some people will only wear two socks if they are exactly the same, but I wouldn’t mind wearing one grey sock and one blue sock. And since the person who posed the problem didn’t specify that he or she insisted that the desired pair of socks had to have the same color, I assume that he or she wouldn’t mind wearing one grey and one blue either.

How many ways are there to shade a total of 8 unit squares in a 4 times 4 grid such that there are exactly 2 shaded squares in each row and each column?

90.

There are 6 ways of shading 2 squares in the top row and 6 ways of shading 2 squares in the next row, for a total of 6*6=36 ways of shading 2 squares in each of the top two rows.

There are now three cases.

First, each column has one shaded square. 6 of the 36 possibilities above fall under this case. You can shade the third row in 6 ways, and then you are forced to shade the fourth row in one way. So this case produces 6*6=36 possible shadings.

Second, two columns have two shaded squares. 6 of the 36 possibilities above fall under this case. You then have only one choice for the third and fourth rows. They both have to be shaded in the columns which had no shaded squares in the first two rows. So this case produces 6*1=6 possible shadings.

Third, one column has two shaded squares, one column has no shaded squares, and the other two columns each have one shaded square. The other 24 of the 36 possibilities above fall under this case. Both the third and fourth row have to have a shaded square in the unshaded column and there are exactly 2 ways of completing the shading. So this case produces 24*2=48 possible shadings.

Altogether there are 36+6+48=90 possible shadings.

A man has 5 female and 7 male friends and his wife has 7 female and 5 male friends. In how many ways can they invite 6 males and 6 females if husband and wife are to invite 6 friends each?

We first assume that none of the 12 people on the husband's list is also on the wife's list. Otherwise, there are different solutions depending on how many of each sex are on both lists.

This problem is perhaps most easily solved by breaking the solutions into cases, depending on how many friends of each sex each spouse selects. There are six possible cases, depending on the number N of females the husband chooses, since N could be 0, 1, 2, 3, 4, or 5. If the husband chooses N of the 5 women, then he must also choose 6-N of the 7 men … and his wife must then choose N of 5 men and 6-N of the 7 women. For each value of N, by the Multiplication Principle of Counting, there are a total of

(5 choose N)*(7 choose (6-N))*(5 choose N)*(7 choose (6-N)) possibilities. (Note that because of the symmetry in the situation described in the problem, each factor is repeated.)

If now you evaluate this for each N from 0 to 5, and then you add the six results together, by the Addition Principle of Counting, you get the following solution:

7^2 + 105^2 + 350^2 + 350^2 + 105^2 + 7^2

How many triangles can be formed by joining any three vertices of a polygon with n sides, the triangles having no sides in common with the polygon?

Let's assume that the polygon is a regular polygon. You want to count the number of ways of picking three vertices. That's easy, it's just "n choose 3".

That's not the correct answer, however, since many of the triangles that are formed from three arbitrary vertices of the polygon will have an edge in common with the polygon.

Well, we could count the number of those triangles, and subtract that number from "n choose 3." For each of the n edges of the polygon you could form n-2 triangles that share that edge, since there are that many left-over vertices that could be used for the third vertex of a triangle … so that there are altogether n*(n-2) of these rejected triangles.

But that's not quite correct since some rejected triangles are counted twice. How can that happen? The only way a triangle would be counted twice is if ithas two edges in common with the polygon — that is, if there are three consecutive vertices A, B, C of the polygon and the triangle is ABC. How many such triangles are there?

Exactly n.

So instead of subtracting n*(n-2), we should subtract n*(n-2) - n — that's the number of rejected triangles — that is, n*(n-3). So the final answer is that there are "n choose 3" - n*(n-3) triangles that have no sides in common with the polygon.

For what values of n is this correct? If n=3, 4, or 5, there are no such triangles, and if n=6, there are exactly 2 such triangles, and these results agree with the expression above, except for n=3. So the reasoning is not quite correct for n=3. Can you find the flaw?

For n=7, there are 6 such triangles, and for n=8, there are 16 such triangles. (Just draw a polygon, number the vertices, and list the "good" triangles.). These agree with the expression above, so the reasoning above is likely correct.

We have thus generated a sequence beginning with 0,0,0,2,7,16. Is this a known sequence? So we go to the Online Encyclopedia of Integer Sequences (oeis.org), an amazing resource, and see if it is there. It turns out that there is a sequence that starts with those six numbers. Is it the same as ours?

Its next term is 32. What is our next term? "9 choose 3" - 9*6 = 84–54 = 30. So this looks like a different sequence. Maybe we made an error. So we draw a 9-sided polygon, number the vertices from 1 to 9, and list all the "good" triangles. Lo and behold, there are exactly 30 of them. So our sequence, the number of "good" triangles in a regular n-gon, is not in OEIS!

It looks like this is a new sequence. The next step is to submit this sequence to OEIS and find out if it is actually a new sequence. You would have thought that someone must have come up with this before. If so, why didn't we find it on OEIS? Perhaps there's a variation of this sequence in OEIS. Or perhaps the expression above is incorrect in general, although it works for n up to 9.

Stay tuned.

Added later: I found it. The sequence is listed, with two 0's instead of three 0's, starting as 0,0,2,7,16,30, … as sequence #A005581, added to OEIS in 2003. It is described there as "the number of inscribable triangles within an (n+4)-gons sharing with them its vertices but not its sides." The expression derived above is modified by combining its two terms to get a "simplified" expression.

Can you find the next number in this sequence: 15, 29, 56, 108, 208,___? What rule did you use?

It is very interesting that all of the responders concluded that the next number was 400, and came up with the same justification for their conclusion.

A similar question was raised in the recent puzzle posed by Will Shortz on his NPR radio program. What is the next number in this sequence: 1, 2, 4, 8, 16, 23, 28, —? And here too there were hundreds of people who submitted that the “next number” was 38, and presumably came up with same justification.

Such problems are inappropriately posed problems because one can justify any number — that’s right, any number — as being the next number after a, b, c, d, — .

That certainly doesn’t sound like a reasonable conclusion, so let me explain why it is correct.

I will start with a simple example. What’s the next number in this sequence: 3, 5, 8, … ?

Is it 10 or 11 or 12 or 13?

You might say that the pattern is “add 2, then add 3, then add 4, etc.” so the next number is 12, and the sequence is 3, 5, 8, 12, 17, 23, …

Or you might say that the pattern is “add 2, then add 3, then add 2, then add 3, etc.” so the next number is 10, and the sequence is 3, 5, 8, 10, 13, 15, 18, …

Or you might say that this looks like the Fibonacci sequence, where every term except the first two is the sum of the two previous terms, so the next number is 13, and the sequence is 3, 5, 8, 13, 21, 34, …

So the next number could be 12, 10, or 13. Could it be 11?

Consider the cubic polynomial $p(n) = -(n^3)/6 + 3(n^2)/2 -4n/3 +3$. It turns out that the terms $p(1)$, $p(2)$, $p(3)$, $p(4)$ are precisely 3, 5, 8, 11, so the sequence could be the sequence of values of the polynomial $p(n)$ for the positive integers n.

This phenomenon happens in general. For the question raised above, for each rational number k, there is a polynomial of degree at most five whose first six values are 15, 29, 56, 108, 208, k !!

So it does not make sense to ask what is “the” next number. It could be anything, depending on what pattern you have in mind. What you are being asked to do is not “find the pattern” but “find the pattern that is in the questioner’s mind.”

That makes questions like this dangerous, particularly since such questions are asked in exams all around the world. If a student sees a pattern that is different from that of the questioner, then he or she is marked wrong … and so is discouraged from finding his or her own patterns in other situations as well. Mathematics becomes a guessing game between the teacher and the student. Very counter-productive.

Questions like this should be discouraged, unless they are formulated as “can you guess my pattern? Can you find another pattern?”

if you are wondering how I constructed that polynomial above, look up the method of finite differences.

The algebraic fact stated above corresponds to the following geometric fact: For each six points in the coordinate plane with rational coefficients, no two of which have the same x-coordinate, there is a polynomial of degree at most 5 with rational coordinates whose graph passes through all six points. (The generalization of these facts is left to you.)

What is the problem attributed to defining if two finite graphs are isomorphic?

There is no problem "defining" when two finite graphs are isomorphic. The problem is that it is not easy to "determine" whether two particular finite graphs are isomorphic.

Try the following activity. Start out with 8 vertices and then without much thought draw enough edges so that there are exactly 3 edges at each vertex. While doing so, be careful not to join two vertices with more than one edge and not to join any vertex to itself. What you have is an example of what is called a 3-regular graph with 8 vertices. Draw three or more additional 3-regular graphs with 8 vertices. Then try to determine whether each pair of graphs is isomorphic.

Use the following definition of isomorphic. Two of these graphs A and B are isomorphic if you can label the vertices of both graphs with the numbers 1,2,3,4,5,6,7,8 so that for every i and j, vertex i and vertex j are joined by an edge in graph A if and only if vertex i and vertex j are joined by an edge in graph B.

This may not be easy to do. In particular, it may be hard to show that two graphs are not isomorphic. Also it can easily happen that two graphs are isomorphic but you can't figure out how to label the vertices so as to demonstrate that they are isomorphic.

To read more about this, find my response to the question "Do mathematicians ever argue about math?"

The question of whether two graphs with any number of vertices are isomorphic is one of the many questions about which it is not known whether there is an algorithm that will always provide the correct answer efficiently. Anyone able to produce such an algorithm, or prove that no such algorithm exists, would win a million dollar prize for having solved the so-called P=NP problem. No one has solved this problem since it was first posed almost 50 years ago, or since the cash prize was announced almost 20 years ago.

How do I take an LCM for large numbers?

The response provided by Kilian Liebe works well, but only when the numbers are relatively small, so that they can be factored efficiently.

Factoring a large number, say one with 40 digits, with any efficiency is nearly impossible. Indeed, when we transmit our credit card information over the internet, their security depends on the essential impossibility of factoring large numbers.

However, even if the numbers can't be factored, you can still find the LCM of two numbers using the following fact: The product of the LCM and the GCD of two numbers equals the product of the two numbers. The proof of this fact is similar to Liebe's solution.

With this fact, we can conclude that the LCM of two numbers is their product divided by their GCD. So to find the LCM of two numbers, we need only find their GCD.

But there is a well known algorithm, known as Euclid's Division Algorithm, for finding the GCD of two numbers. I'll illustrate the algorithm with an example.

Suppose you want to find the GCD of 2329 and 5117. You divide the smaller into the larger and get that 5117 = (2 x 2329) + 459 — that is 2329 goes into 5117 twice, with a remainder of 459.

Now you repeat this division with 2329 and 459: 2329 = (5 x 459) + 34.

Now you repeat this division with 459 and 34: 459 = (13 x 34) + 17.

Once more: 34 = 2 x 17.

We have found that 17 divides 34, and therefore 17 divides 459 (since it divides both 17 and 34), and therefore 17 divides 2329 (since it divides both 34 and 459), and therefore 17 divides 5117 (since it divides both 459 and 2329).

So 17 is a common divisor of 2329 and 5117.

Suppose, on the other hand, that N was a common factor of 2329 and 5117. Then it would also have to be a factor of 459, and therefore of 34, and therefore of 17. So any common divisor of the two original numbers is a factor of 17.

So 17 is the GCD of 2329 and 5117.

This method of finding the GCD works for any two positive integers, and so we can also find the LCM of the original two integers. For example, the LCM of 2329 and 5117 is (2329 x 5117) / 17

Notice that we have not factored 2329 or 5117 — we have just found the GCD, the largest factor that they share.

If you wanted to find the LCM of three numbers, you could find the LCM of the first two numbers and then find the LCM of that and the third number, etc.

How does mathematical induction work?

Imagine a ladder that goes up forever and you want to climb this ladder. You know how to get from the ground to the first rung (=step) of the ladder (first method) and you also how to get from any rung to the next rung (second method). Then you can climb up to any rung of the ladder.

How do you do that? You use the first method to get to the first rung, then the second method to get from the first rung to the second rung, then the second method again to get from the second rung to the third rung, then the second method again to get from the third rung to the fourth rung, and so on, until you get to your desired rung.

That's how mathematical induction works. The basis step — the first method — gets you off the ground to the first rung, and the induction step— the second method — allows you to get from any rung to the next rung. The conclusion is that these two methods enable you to get to any step of the ladder, that is, you can conclude that P(n) is true for all n.

If there are 5 people, what is the probability that at least two of them have been born in the same month? I have tried it as ((5C2)*12^4)/12^5 as we can choose any of them to be together in any month and the rest of the people can be in any month.

Suppose that people #1 and #2 are born in January, that #3 and #4 are born in February, and #5 is born in March. Then this outcome would be counted twice in your calculation, once for the first two people and once for the next two. Even worse, if all five are born in January, then that outcome would be counted no less than 5C2=10 times. So your formula overcounts the good cases, where at least two are born in the same month.

This question, like another one I answered recently, can be handled using the complementary event. Instead of calculating directly the probability of an event E, you calculate rhe probability of the complementary event E'. What is the complementary event? Remember that an event E is a set of outcomes; the complementary event E' consists of all outcomes that are not in E. For example, if you toss a die, and E is the event that a prime number comes up, then E={2,3,5} and E'={1,4,6}?.

The critical fact is that P(E)=1-P(E') — make sure you understand why that is true — so that if know the probability of the complementary event E', you can use this formula to find the probability of E.

In the given situation, E is all outcomes in which at least two of the five people were born in the same month. Then the complementary event E' consists of all outcomes where all five people were born in different months. Then P(E') is easy to calculate. It is just P(E')= 12*11*10*9*8/12^5 = 55/144, and therefore P(E) = 1 - 55/144 = 89/144.

Many probability problems can be solved using this method.

How many onto functions are there from A to B if n(A) = m and n(B) = n?

Another question to which I submitted an answer previously was exactly this question for m=4 and n=3. That is, how many onto functions are there from {a,b,c,d} to {1,2,3}?

Let’s keep n=3 and ask how many onto functions are there from an m-element set M to {1,2,3}?

Any such function maps i elements of M to 1, j elements of M to 2, and k elements of M to 3, where i, j, and k are all positive, and their sum is m.

The number of such functions, for fixed i, j, k, is “m choose i” times “m-i choose j”, by the Multiplication Principle of Counting.

You have to take the sum of these expressions over all ordered pairs <i,j> for which i and j are positive and i + j < m, by the Addition Principle of Counting.

If m=3, then there is only one ordered pair <1,1>, so we get “3 choose 1” times “2 choose 1” or 6.

If m=4, then there are three ordered pairs <1,1>, <2,1>, and <1,2>, so we get 12 + 12 + 12 = 36.

If m=5, then there are six ordered pairs <1,1>, <2,1>, <1,2>, <3,1>, <2,2>, and <1,3>, so we get 20 + 30 + 30 + 20 + 30 + 20 = 150.

If m=6, then there are ten ordered pairs, the six above plus <4,1>, <3,2>, 2,3> and <4,1>, so we get 30 + 60 + 60 + 60 + 90 + 60 + 60 + 60 + 60 + 60 = 600.

Is there a pattern here? The first four terms, for m=3,4,5,6, are 6, 36, 150, 600. Can we guess what the next term should be?

At this point, the best move is to check the On-line Encyclopedia of Integer Sequences at The On-Line Encyclopedia of Integer Sequences® (OEIS®) and see whether our sequence is among the thousands and thousands of sequences listed there.

Sad news. It’s not there. That probably means that there is no simple formula that will generate the answers to the questions of how many onto functions there are from an m-element set to a three element set. If that’s the case, then there is also no formula that will tell us how many onto finctions there are from an m-element set to an n-element set.

However, we described above a method that will give us the answer for any particular m and n, and even a way of writing a program that will churn out the answer.

Added later: The total given here for the number of onto functions from a 6-element set to a 3-element set is incorrect. It should be 540, not 600. The total given for the <4,1> case should be 30 instead of 60, as should be the total for the <1,4> case. As a result, the sequence becomes 6, 36, 150, 540, which is the beginning of a sequence in the Online Encyclopedia of Integer Sequences, and you can find lots of information about this at OEIS.org

In how many ways can a class of 22 students be split into two groups of 9 and 13 respectively, if there is a pair of twins who must not be separated?

Forget the twins for a moment, and think of the class as having just 20 students. Now you can pick any 7 students from the class and add the twins to that group. Or you can pick any 9 of the students from the class and add the twins to the other group.

In either case you get one group of 9 students and one group of 13 students. Moreover, any way of splitting the class into two groups of the specified sizes where the twins are together falls into one of these two categories.

Thus, altogether, the number of ways is “20 choose 7” plus “20 choose 9”.

Added later: Look at my answer from the perspective of Pascal’s Triangle T. On the 20th row of T, three consecutive entries are “20 choose 7”, “20 choose 8”, and “20 choose 9”. If you add the first two you get “21 choose 8” and if you add the last two, you get “21 choose 9”. If you add those together you get “22 choose 9”.

The answer in my previous comment was “20 choose 7” plus “20 choose 9”. If you follow the arithmetic in the paragraph above, you find that the previous answer equals “22 choose 9” minus twice “20 choose 8”.

So there should be a line of reasoning that will lead to that answer. Indeed there is! You could choose any 9 of the 22 students and then reject those groups of 9 which contain one of the twins (“20 choose 8” of them) and also reject those groups of 9 which contain the other of the twins (another “20 choose 8” of them).

What is a combinatorial argument to that the sum of the first n odd square is $(2n+13)(2n+13)$?

Look at Pascal’s Triangle. The left-most diagonal consists entirely of 1s. The next diagonal contains all the counting numbers. The next diagonal contains all the “choose 2” numbers, that is all the number of the form “n choose 2”, the number of ways of choosing 2 out of n objects. These numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, … For this reason, this diagonal is called the 2′nd diagonal of Pascal’s Triangle, the previous diagonal is called the 1’st diagonal, and the diagonal of 1s at the left is called the 0′th diagonal of Pascal’s Triangle. The next diagonal, the 3rd diagonal, contains the “choose 3” numbers, and so on.

If you add the choose 2 numbers, 1+3+6+10+15+21+28+36+45, from “2 choose 2” to “10 choose 2”, you get 165, which happens to be 11 choose 3, 11x10x9/3x2x1. In general, if you add the “choose m” numbers from “m choose m” to “n choose m”, you get “n+1 choose m+1”. This is one of the amazing features of Pascal’s Triangle.

What does all of this have to do with the question of finding the sum of the first n odd square numbers?

Let's see. You will notice that the sum of two consecutive "choose 2" numbers is a square number, and that if you start with 1, you get 1, 4, 9, 16, 25, 3, 49, 64, 81, … . But if you group the "choose 2 numbers" as {1}, {3,6}, {10,15}, {21, 28}. {36,45}, … you see that the sum of the "choose 2" numbers 1+(3+6)+(10+15)+(21+28)+(36+45) is the same as the sum of the first 5 odd squares. This is also, as we have seen above "11 choose 3". Thus the sum of the first 5 odd squares is "11 choose 3".

The same reasoning works in general, and yields the indicated conclusion: The sum of the first n odd squares is "2n+1 choose 3".

How many factors does 100^100 have? Answer: 201*201=40401 (including 1) But how? I've not found it on Google.

Since 100 = 2*2*5*5, 100^100 = (2^200)*(5^200).

Any factor of 100^100 must therefore be a power of 2 times a power of 5.

What powers of 2 are possible? Any integer from 0 to 200, so there are 201 possible powers of 2 in a given factor of 100^100. Similarly, there are 201 possible powers of 5 in a given factor of 100^100.

The Multiplication Principle of Counting says that if there are A ways to do one task and there are B ways of doing a second task, and the tasks are independent of each other, then the total number of ways of doing both tasks is A*B. For example, if you have five shirts and three pairs of pants, you can make altogether 5*3=15 possible outfits (although you may not want to wear some of them) since any shirt can go with any pair of pants.

Back to our problem, since any factor of 100^100 is like an outfit consisting of a power of 2 together with a power of 5, and there are 201 possible powers of 2 and 201 possible powers of 5, the total number of outfits, oops, factors is 201*201.

Of course, the Multiplication Principle of Counting is true when there are any number of tasks and so a similar analysis will tell you how many factors any whole number N has, so long as N is factored into prime numbers (which may itself be a big challenge).